0.1 Basic formulation for plates and shells

0.1.1 Some assumptions for the kinematic model of the plate

A necessary condition for applying the plate/shell model framework to a deformable body is that a geometrical midsurface might be, if only loosely, recognized for such a body. Then, an iterative refinement procedure[1] may be applied to such tentative midsurface guess.

Then, material should be observed as *[piecewise-]*homogeneous, or slowly varying in mechanical properties while moving at a fixed distance from the midsurface.

Of the two outer surfaces, one has to be defined as the *upper* or *top* surface, whereas the other is named lower ot *bottom*, thus implicitly orienting the midsurface normal towards the top.

Finally, the body should result fully determined based on a) its midsurface, b) its pointwise thickness, and c) the through-thickness (TT) distribution of the constituent materials.

The geometrical midsurface is of little significance if the material distribution is not symmetric[2]; such midsurface, in fact, exhibits no relevant properties in general. Its definition is nevertheless pretty straighforward.

In the present treatise, a more general *reference* surface definition is preferred to its median geometric counterpart; in particular, an *offset* term o is considered that pointwisely shifts the geometric midsurface with respect to the reference surface. A positive offset shifts the midsurface towards the top.

With the introduction of the offset term, the reference surface may be arbitrarily positioned with respect to the body itself; as an example, an offset set equal to plus or minus half the thickness makes the reference surface correspondent to the bottom or top surfaces, respectively.

Such offset term becomes fundamental in the Finite Element (FE) shell implementation, where, in fact, the reference plane is uniquely

[1]Normal segments may be cast from each point along the midsurface, that end on the outer body surfaces. The midpoint locus of these segments redefines the midsurface itself.

[2]If the unsimmetric laminate is composed by isotropic layers, a reference plane may be obtained for which the $\underline{\underline{B}}$ membrane-to-bending coupling matrix vanishes; a similar condition may not be verified in the presence of orthotropic layers.

defined by the position of the nodes, whereas the offset arbitrarily shifts the geometrical midsurface.

In the case of limited[3] curvatures, and for considerations whose scope is local, the tangent reference plane may be employed in place of the possibly curve reference surface, thus locally reducing the general shell treatise to its planar, plate counterpart.

Figure 1 shows the basic kinematic relations for the shear deformable (Mindlin) plate model; in the undeformed configuration, P is a generic material point along the plate thickness, and Q is its normal projection on the reference plane. Such Q point is named *reference point* for the TT normal segment it belongs to.

A local reference system is defined, whose third axis z is normal to the undeformed midsurface; the first in-plane (IP) x axis may be arbitrarily oriented, e.g. by projecting a global $\hat{v}$ unit vector, and the remaining y axis may be construed such that it finalizes the right xyz triad.

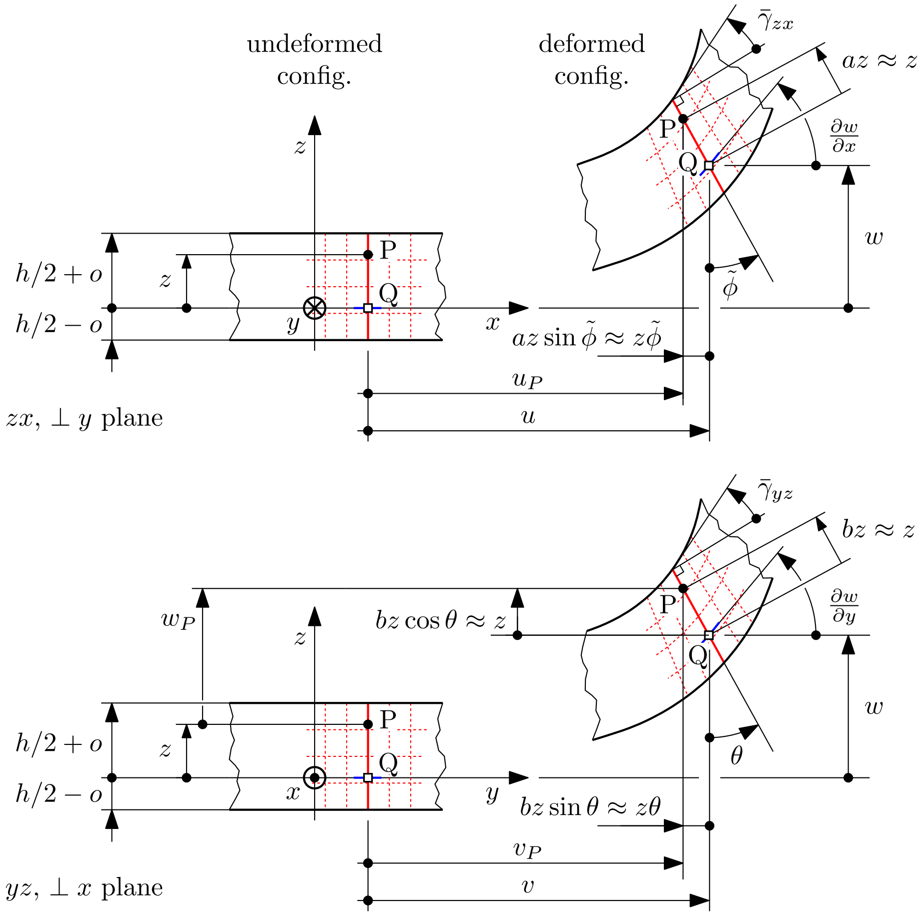
Then, the deformed configuration is considered, and the motion of both the points is monitored according to two mutually orthogonal views.

The P displacement components $(u_\text{P}, v_\text{P}, w_\text{P})$ may be defined as a function of the motion of its reference point Q, described in terms of its displacement components (u, v, w), plus the two θ, ϕ rotation components with respect to the x, y IP local axes, respectively. Those angular displacements are defined with respect to the normal segment orientation, as measured on the orthogonally projected views. After some cumbersome trigonometric manipulations[4] we obtain

$$\begin{aligned}
u_P &= u + z\,(1+\check{\epsilon}_z)\,\frac{\cos\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}}\sin\phi \\
v_P &= v - z\,(1+\check{\epsilon}_z)\,\frac{\cos\phi}{\sqrt{1-\sin^2\phi\sin^2\theta}}\sin\theta \\
w_P &= w + z\left((1+\check{\epsilon}_z)\,\frac{\cos\phi\cos\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}}-1\right),
\end{aligned}$$

[3] with respect to thickness

[4] in which it may happen to miss some higher order terms, as the author persistently did in previous versions of the present notes



$$\{a, b\} = \frac{(1 + \check{\epsilon})}{\sqrt{1 - \sin^2\phi \sin^2\theta}} \{\cos\theta, \cos\phi\} \approx 1$$

Figure 1: Relevant dimensions for describing the deformable plate kinematics. Here, two a, b factors are introduced which reduce to unity for small rotations and strain.

where $z\,(1+\check{\epsilon}_z)$ is the length of the PQ segment on the deformed configuration, which is further scaled by the fractional factors due to projection along Fig. 1 views.

The $\check{\epsilon}_z$ average z strain term is defined based on the accumulation of the Poisson shrinkage (or elongation) along the PQ segment, i.e.

$$\begin{aligned}\check{\epsilon}_z(z) &= \frac{1}{z}\int_0^z \epsilon_z d\varsigma\\ &= \frac{1}{z}\int_0^z -\frac{\nu}{1-\nu}\left(\epsilon_x+\epsilon_y\right)d\varsigma,\end{aligned}$$

the second expression holding in the case of isotropic materials only.

The stress component σ_z which is normal to the reference surface is in fact assumed to be either zero or negligible. Being a full discussion[5] of such a plane stress assumption beyond the scope of the present contribution (BSPC), we limit our treatise to the observation that, in the inevitably anecdotal case of Fig. 2, the ratio between the OOP σ_{z} stress component and its IP counterparts varies with the square of the ratio between the thickness and an in plane significant length. The engineering relevance of such a normal stress component rapidly vanishes with increasing plate thinness. The Fig. 2 examples also points out the intermediate magnitude decay of the OOP shear stresses, whose normalized form linearly varies with the same thinness ratio.

Such displacement components may be linarized with respect to i) the small rotations and ii) small ϵ_z strain hypotheses, thus obtaining the following expressions

$$u_P = u + z\phi \tag{1}$$

$$v_P = v - z\theta \tag{2}$$

$$w_P = w. \tag{3}$$

[5]Such assumption is coherent with the free surface conditions at the top and the bottom skins, and with the moderate thickness of the elastic body, that allows only a narrow deviation from the boundary values. In fact, the equilibrium of a partitioned, TT material segment requires that

$$\sigma_z(z) = -\int_{-h/2+o}^{z} \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y}dz = +\int_{z}^{+h/2-o} \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y}dz,$$

where τ_{zx}, τ_{yz} are the interlaminar, OOP shear stress components, whose IP gradient is limited.

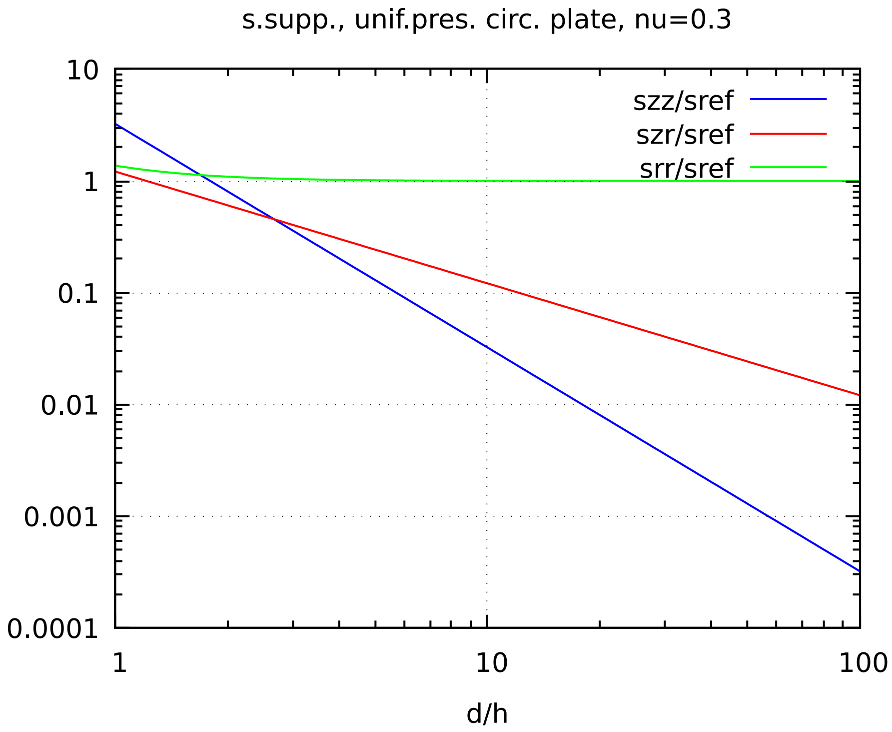


Figure 2: Normalized stress component magnitude in the case of a simply supported circular plate subject to normal pressure, according to the spatial theory of elasticity framework, see [1, p.349]. A homogeneous and isotropically elastic circular plate of diameter d and thickness h is simply supported along its perimeter (i.e. apart from the their transverse component, displacements are free, and so are rotations), and it is loaded by a unit pressure at its upper surface. The peak magnitude of the transverse stress σ_{z} is observed at the pressurized surface, and it equates the pressure value. The OOP shear stress τ_{zr} is maximal along the perimeter, and it equates $\frac{3}{8}\left(\frac{d}{h}\right)$. The two equal IP direct stress components $\sigma_{\mathrm{r}} = \sigma_{\theta}$ reach the peak value of $\frac{3(\nu+3)}{32}\left(\frac{d}{h}\right)^2 + \frac{\nu+2}{20}$ in correspondence of the plate center, at the surface; its thin plate counterpart, σ_{ref}, which lacks the second term, is taken as the normalizing stress value. The remaining $\tau_{\mathrm{r}\theta}$, $\tau_{\theta\mathrm{z}}$ stress components are zero due to axisymmetry. The commonwise $\nu = 0.3$ Poisson ratio value is used in tracing the Figure.

A treatise of the large rotation and/or large strain nonlinear case is, again, BSPC.

According to such linearized expression, the kinematics of the P points originally[6] laying on a TT segment that is normal at Q to the reference surface may be described as that of a rigid body.

The intrinsic shear related warping is either negated or neglected, along with any sliding motion of the P points along the segment[7].

Also, the behaviour of such a segment is coherent with its rigid body modeling from the external loads point of view; in particular the external actions act on the plate deformable body only through their TT resultants, and no stress/strain components, or work, are associated by the shell framework to wall squeezing actions, e.g. laminations.

We thus observe that, according to the shell framework, the following external actions are not distinguishable: i) a q pressure applied at the upper surface, ii) a $-q$ traction applied at the lower surface, iii) a q differential pressure between the outer surfaces, with $p+q$ applied at the top, and a generic p applied at the bottom, and iv) a transverse inertial force whose area density is q, namely due to a oppositely oriented $\frac{q}{\rho h}$ acceleration, where ρ is the material density. Also, a fp, friction induced, x-oriented shear action at the upper surface is not distinguishable from an analogous distributed force for unit area applied at the reference surface, plus a y-oriented distributed moment per unit area, whose magnitude is $fp(h/2+o)$.

By observing the deformed configurations in Fig. 1, the normal displacement $\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)$ gradient – i.e. the gained slope of the deformed reference surface, with respect to its original orientation – is made up of two terms, namely the rotation of the normal segment, which originates from the accumulation of the flexural curvature, and the shear compliance, which resembles the transverse slippage typical of a card deck. The following expressions are derived

[6] i.e. in the undeformed configuration

[7] The elision of higher order terms renders the laminate kinematically – but not elastically – indistinguishable from its counterpart that might derive from a plane *strain* assumption.

$$\frac{\partial w}{\partial x} = \bar{\gamma}_{zx} - \phi \tag{4}$$

$$\frac{\partial w}{\partial y} = \bar{\gamma}_{yz} + \theta \tag{5}$$

in which the bar notation employed for the OOP shear components emphasizes their TT average nature.

0.1.2 Local and generalized strains

The IP strain components may hence be derived at the P point through differentiation, and in particular we have

$$\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \tag{6}$$

$$\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial \theta}{\partial y} \tag{7}$$

$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{8}$$

$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x}\right) \tag{9}$$

It clearly appears from the expressions above that the pointwise strain values are due to the sum of i) the strain components as observed at the reference plane,

$$\underline{\mathrm{e}} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{bmatrix} \equiv \underline{\epsilon}_{\mathrm{Q}} \tag{10}$$

which are named *membrane* strains[8] in the customary case in which the material is symmetric[9] with respect to the reference plane, plus ii) terms that linearly scale with the z distance from such a plane, whose

[8] $\underline{\mathrm{e}}$ is an alternative symbol for the more natural, and previously employed $\bar{\underline{\epsilon}}$, whose double barred appearance is however terrible.

[9] or, more generally, elastically balanced

coefficients

$$\underline{\kappa} = \begin{bmatrix} +\frac{\partial \phi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x} \end{bmatrix} = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{11}$$

are named *curvatures*.[10] The strains at the reference surface, and the curvatures constitute the set of plate [shell] *generalized strain components*, which are e.g. usually returned by Finite Element (FE) solvers; those components allow for the following compact representation of the IP strains at P

$$\underline{\epsilon}_{\mathrm{P}} \equiv \underline{\epsilon} = \underline{\mathrm{e}} + z\,\underline{\kappa}\,. \tag{12}$$

It worth to be stressed that the kinematic assumptions for the plate model impose a linear TT profile for each single IP strain component; those components may hence be sampled at the outer surfaces alone, without loss of information. It is here anticipated that an analogous behaviour is proper of the IP stress components if and only if (IIF) the material is elastically homogeneous along the thickness .

The two κ_x and κ_y curvatures equate to the inverse of the normal curvature radii, as probed along the respective local directions; those curvatures are positive if the upper plate fibers are stretched, or, equivalently, if the reference surface acquires convexity if observed from above – i.e. from a point on the positive z axis.

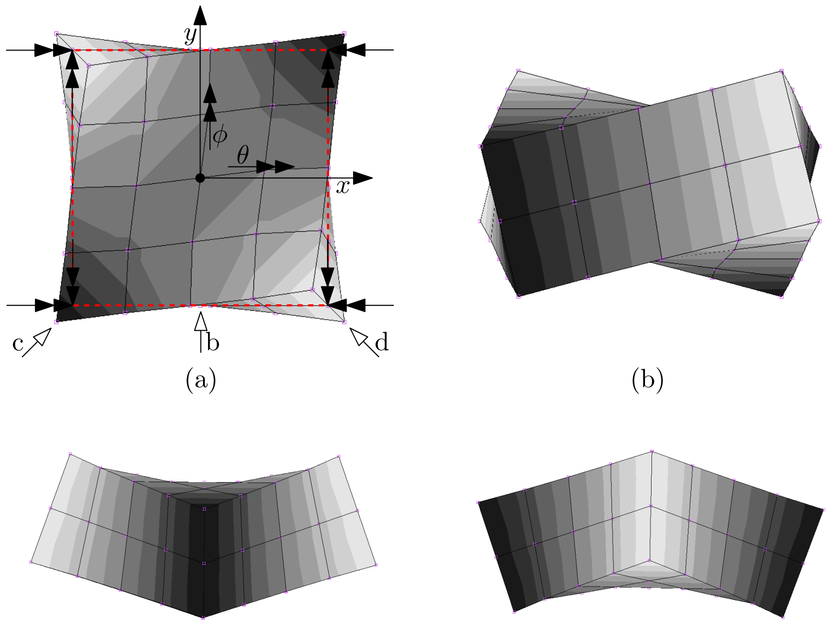
Figure 3 clarifies the nature of the *mixed* curvature term κ_{xy}, which is e.g. typical of open thin walled members – and flat plates as a particular case – subject to torsion[11].

[10]Please note that in the case of shells, the bare *curvature* name may be confusing, since it might refer to either

- the initial, original, geometric, undeformed curvature, which is proper of the shell before the application of some external loads, or to the
- strain, strain-induced, elastic[-plastic], bending, flexural curvature, or curvature change, which consist in the variation of the thin wall curvature due to the effect of the applied loads.

Except for [locally] flat panels, the author suggests to always specify which kind of curvature we refer to. Here, *curvature* is used with reference to *curvature change*.

[11]the *torsional* curvature denomination for the κ_{xy} term, that the present author has widely employed in the past, is not so proper nor widespread, so it might be better avoided. Flexure and torsion are in fact not as uncoupled in the plate realm as they are in beam theory, and *flexure* might be conveniently employed as an



(a)

(b)

(c)

(d)

Figure 3: Positive κ_{xy} mixed curvature for the plate element. The grayscale coloring is proportional to the normal displacement w, which spans from an extremal downward deflection (black), to an equal in modulus extremal upward deflection (white). The gray level at the centroid is associated to zero. Subfigure (a) shows the positive γ_{xy} shear strain at the upper surface, the IP undeformed midsurface, and the negative γ_{xy} at the lower surface; the point of sight related to subfigures (b) to (d) are also evidenced. θ and ϕ rotation components decrease with x and increase with y, respectively, thus leading to positive κ_{xy} contributions. As shown in subfigures (c) and (d), the mixed curvature of subfigure (b) evolves into two anticlastic bending curvatures if the reference system is aligned with the square plate element diagonals, and hence rotated by 45° with respect to z.

0.1.3 Stresses, and their through-thickness resultants

The IP stress components at P are derived from strains by referring to the material elastic constants, and to the plane stress hypothesis. We hence have

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} = \underline{\underline{D}}\,\underline{e} + z\,\underline{\underline{D}}\,\underline{\kappa}\,, \tag{13}$$

where $\underline{\underline{D}}$ embodies the material constitutive law which elastically relates to IP stress/strain components, and which is derived according to the plane stress hypothesis.

In the particular case of an isotropic material – the generally orthotropic case is treated below – such a matrix takes the form

$$\underline{\underline{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \tag{14}$$

whereas the normal component of strain, which is due to the Poisson shrinkage alone, may be evaluated as

$$\epsilon_z = -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right). \tag{15}$$

The attentive reader may observe that no mention is made to the OOP shear stresses, to which a paragraph is devoted below.

Moreover, the absence of transverse shear terms in current paragraph formulation, and in particular in Eq. 13, hints for the IP and the OOP stress/strain components to be elastically uncoupled; the material has evidently been *implicitly* assumed as monoclinic with respect to the reference surface. Such a condition holds e.g. for isotropic materials, and for the orthotropic plies usually employed in laminates.

As in the classical theory of beams, stress components are integrated along the relevant unit of analysis, namely the cross section there, and the normal segment here, to obtain suitable internal action resultants.

umbrella term that also encompass profile (open and thin) wall deformation due to pure torsion.

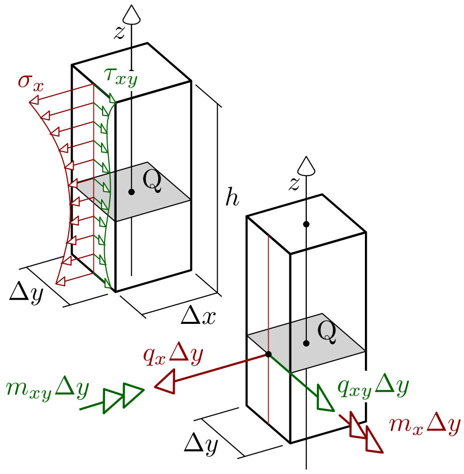


Figure 4: XXX

According to the thin plate framework, stress resultants take the form of forces per unit length along the surface, and they may be expressed as

$$\underline{q} = \begin{bmatrix} q_x \\ q_y \\ q_{xy} \end{bmatrix} = \int_h \underline{\sigma}\, dz$$
$$= \underbrace{\int_h \underline{\underline{D}}\, dz}_{\underline{\underline{A}}}\ \underline{e} + \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}}}\ \underline{\kappa} \tag{16}$$

in the case of the IP components, whereas for the OOP components we have

$$q_{xz} = \int_h \tau_{zx} dz \qquad\qquad q_{yz} = \int_h \tau_{yz} dz. \tag{17}$$

Those quantities may be interpreted with respect to their (doubled if single) subscripts as follows: q_{ab} is the b component of internal action that is transmitted through a TT imaginary gate, whose in plane width is unit and whose normal is oriented along a. According to this rationalization, the $\underline{q}$ components are also called *stress flows*.

Besides the internal action resultants of the force kind, by weighting the stress component contribution based on their z lever arm we obtain the moment stress resultants (or *moment flows*), whose expressions follow

$$\underline{\mathrm{m}} = \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} = \int_h \underline{\sigma}\, z dz$$
$$= \underbrace{\int_h \underline{\underline{\mathrm{D}}}\, z dz}_{\underline{\underline{\mathrm{B}}} \equiv \underline{\underline{\mathrm{B}}}^{\mathrm{T}}}\ \underline{\mathrm{e}} + \underbrace{\int_h \underline{\underline{\mathrm{D}}}\, z^2 dz}_{\underline{\underline{\mathrm{C}}}}\ \underline{\kappa}\,. \tag{18}$$

A selection of internal action components is represented in Fig. 4 shows, along with the stress distributions they arise from.

0.1.4 Constitutive equations for the plate

By employing the matrices defined in Eqs. 16 and 18, the cumulative generalized strain - stress resultants relations for the plate (or for the laminate) may be summarized in the following expressions

$$\begin{bmatrix} \underline{\mathrm{q}} \\ \underline{\mathrm{m}} \end{bmatrix} = \begin{bmatrix} \underline{\underline{\mathrm{A}}} & \underline{\underline{\mathrm{B}}} \\ \underline{\underline{\mathrm{B}}}^{\mathrm{T}} & \underline{\underline{\mathrm{C}}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{e}} \\ \underline{\kappa} \end{bmatrix} \tag{19}$$

which are usually referred to as the *constitutive equations* of the [laminate] plate, and the coefficient matrix, named *constitutive matrix* for the laminate, summarizes the elastic response of the latter.

The contribution of the IP stress/strain components to the elastic strain energy area density[12] is defined based on the previous relation as

$$\upsilon^{\dagger} = \frac{1}{2} \begin{bmatrix} \underline{\mathrm{q}} \\ \underline{\mathrm{m}} \end{bmatrix}^{\top} \begin{bmatrix} \underline{\mathrm{e}} \\ \underline{\kappa} \end{bmatrix} \tag{20}$$
$$= \frac{1}{2} \begin{bmatrix} \underline{\mathrm{e}} \\ \underline{\kappa} \end{bmatrix}^{\top} \begin{bmatrix} \underline{\underline{\mathrm{A}}} & \underline{\underline{\mathrm{B}}} \\ \underline{\underline{\mathrm{B}}}^{\mathrm{T}} & \underline{\underline{\mathrm{C}}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{e}} \\ \underline{\kappa} \end{bmatrix}. \tag{21}$$

The $\underline{\underline{\mathrm{A}}}$ and the $\underline{\underline{\mathrm{C}}}$ minors of the constitutive matrix characterize the plate stiffness with respect to membrane and flexural load case families respectively; the membrane/flexural coupling stiffness minor

[12] i.e. strain energy per unit reference surface area

$\underline{\underline{\mathrm{B}}}$, which is in general nonzero, vanishes if the material is symmetrically distributed with respect to the reference surface.

In the commonwise case of TT homogeneous material, and null offset[13] we have

$$\underline{\underline{\mathrm{A}}} = h\,\underline{\underline{\mathrm{D}}} \qquad \underline{\underline{\mathrm{B}}} = \underline{\underline{0}} \qquad \underline{\underline{\mathrm{C}}} = \frac{h^3}{12}\underline{\underline{\mathrm{D}}},$$

i.e. the membrane stiffness varies linearly with the wall thickness, the flexural stiffness varies with the cube of the thickness, and the membrane and the flexural loadings are mutually uncoupled. Such a laminate elastic properties dependence on thickness essentially holds also for laminates, if the TT distribution of the various materials is kept comparable.

0.1.5 The transverse shear stress/strain components

A full treatise on the title topic is, due to its complexity, BSPC; starting points for further investigation my be found in [2], [3] or in the theory manual of your favourite FE solver.

The two $\bar{\gamma}_{yz}$ and $\bar{\gamma}_{zx}$ transverse shear components are in fact more directly recognizable as further contributions to the $\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)$ normal deflection gradient, with respect to what is attributable to flexure alone, than TT averages of actual, pointwise shear strains – see e.g. Figure 1.

Also, the two q_{xz}, q_{yz} stress flow components defined in Eq. 17 are recognized to perform work[14] on the same $\bar{\gamma}_{yz}$ and $\bar{\gamma}_{zx}$ transverse shear components, respectively; the transverse shear contribution to the elastic strain energy per unit ref. surface area is hence

$$\upsilon^{\ddagger} = \frac{1}{2} q_{xz}\bar{\gamma}_{xz} + \frac{1}{2} q_{yz}\bar{\gamma}_{yz}. \tag{22}$$

The constitutive equation for the transverse shear is set at normal segment (vs. punctual) level, with the declared aim of collecting the

[13] In the presence of a nonzero offset between the reference and the median planes, the uncoupled nature of the plate membrane/flexural loadings is only *formally* lost. If the same problem is considered based on a median reference plane, in fact, such a property is obviously restored.

[14] in particular, work for unit reference surface area

elastic strain energy contributions along the thickness, and they are usually formulated as

$$\upsilon^{\ddagger} = \frac{1}{2} \begin{bmatrix} \bar{\gamma}_{xz} \\ \bar{\gamma}_{yz} \end{bmatrix}^{\top} \underbrace{\chi \int_h \underline{\underline{\mathrm{G}}}\, dz}_{\underline{\underline{\Gamma}}} \begin{bmatrix} \bar{\gamma}_{xz} \\ \bar{\gamma}_{yz} \end{bmatrix}, \tag{23}$$

where $\underline{\underline{\mathrm{G}}}$ is the pointwise constitutive matrix for the transverse shear components[15] – which is considered through its TT integral, χ is a *shear correction factor* – which accommodates for possibly any incongruence in the formulation, and $\underline{\underline{\Gamma}}$ is an emended transverse shear constitutive matrix for the whole plate. By comparing Eqns. 22 and 23 we also derive the *de facto* transverse shear constitutive relation

$$\begin{bmatrix} q_{xz} \\ q_{yz} \end{bmatrix} = \underline{\underline{\Gamma}} \begin{bmatrix} \bar{\gamma}_{xz} \\ \bar{\gamma}_{yz} \end{bmatrix}. \tag{24}$$

for the Mindlin shear deformable plate.

In the case of isotropic materials, $\underline{\underline{\mathrm{G}}}$ is a diagonal matrix whose terms equate the shear modulus, i.e.

$$\underline{\underline{\mathrm{G}}} = \frac{E}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

whereas the χ shear correction factor is usually assumed as $\frac{5}{6}$ if the material is TT uniform[16]; different χ values are however proposed in literature, see e.g. [4], along with different procedures[17] for evaluating $\underline{\underline{\Gamma}}$.

In the case pointwise values are requested for the τ_{zx} and τ_{yz} stress components – e.g. in the analysis of interlaminar stresses in composite laminates, those quantities are derived from the assumed absence of

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[15] $\underline{\underline{\mathrm{G}}}$ is the 2 by 2 matrix s.t. $\begin{bmatrix} \tau_{zx} \\ \tau_{yz} \end{bmatrix} = \underline{\underline{\mathrm{G}}} \begin{bmatrix} \gamma_{zx} \\ \gamma_{yz} \end{bmatrix}$.

[16] please note the parallel with the inverse 1.2 correction factor for the shear contribution to the beam elastic strain energy, proper of the solid rectangular cross section.

[17] we report as an example the notable case of of honeycomb panels – whose transverse shear compliance is rarely negligible, in which Γ is defined as the $\underline{\underline{\mathrm{G}}}_{\text{foam}}$ transverse shear constitutive matrix for the foam/honeycomb material interposed between the outer skins, multiplied by the overall panel thickness h; in this case the χ transverse shear correction factor is implicitly defined as unity.

shear stresses on the lower surface, and by accumulating the IP stress component contributions to the x and y translational equilibria up to the desired z sampling height. We hence obtain

$$\tau_{zx}(z) = -\int_{-\frac{h}{2}+o}^{z} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} dz \tag{25}$$

$$\tau_{yz}(z) = -\int_{-\frac{h}{2}+o}^{z} \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} dz. \tag{26}$$

The parallel is evident with the Jourawsky theory of shear for beams.

0.1.6 Hooke's law for the orthotropic lamina

Hooke's law for the orthotropic material IP stress conditions, with respect to principal axes of orthotropy;

$$\underline{\underline{D}}_{123} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \underline{\underline{T}}_1 \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \underline{\underline{T}}_2 \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{28}$$

where

$$\underline{\underline{T}}_1 = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \tag{29}$$

$$\underline{\underline{T}}_2 = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \tag{30}$$

α is the angle between 1 and x;

$$m = \cos(\alpha) \qquad n = \sin(\alpha) \tag{31}$$

The inverse transformations may be obtained based on the relations

$$\underline{\underline{T}}_1^{-1}(+\alpha) = \underline{\underline{T}}_1(-\alpha) \qquad \underline{\underline{T}}_2^{-1}(+\alpha) = \underline{\underline{T}}_2(-\alpha) \tag{32}$$

Finally

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} \qquad\qquad \underline{\underline{D}} \equiv \underline{\underline{D}}_{xyz} = \underline{\underline{T}}_1^{-1}\,\underline{\underline{D}}_{123}\,\underline{\underline{T}}_2 \tag{33}$$

With regard to the transverse shear constitutive relation, in the case of an orthotropic material whose OOP shear moduli are G_{z1} and G_{2z} we have

$$\underline{\underline{G}} = \begin{bmatrix} n^2 G_{z1} + m^2 G_{2z} & mnG_{z1} - mnG_{2z} \\ mnG_{z1} - mnG_{2z} & m^2 G_{z1} + n^2 G_{2z} \end{bmatrix}.$$

0.1.7 An application: the four point bending test specimen.

The case of the four point bending test is considered, see Figure 5a, with an isotropic and homogeneous specimen material. Specimen dimensions are defined as in Figure, where the b the specimen width is taken as the relevant unit of length.

The width to length ratio of the specimen is less than unity, but far from being negligible; a treatise according to the plate theory would hence be more appropriate than the beam model which is usually proposed by normative.

Such a test is based on the assumption that the bending moment – a beam framework quantity – is constant along the gauge length, and equal to Fl; such a quantity equates the through-width (TW) integral of the m_{x} moment resultant, whose value is assumed TW constant and equal to $m_{\mathrm{x}}^* = Fl/b$. The specimen curvature along the gauge length is

$$k_{\mathrm{x}}^* = \frac{12Fl}{Ebh^3} \tag{34}$$

according to the beam theory; such a value taken as a reference.

The treatise according to the plate theory is far less straightforward that its trivial beam counterpart, since e.g. we may consider the two opposite extremal cases of i) unconstrained anticlastic secondary curvature, or, equivalently, null m_{y} transverse (in the sense of TW, not TT) moment resultant, and ii) cylindrical bending, i.e. null transverse κ_{y} curvature. The membrane generalized stress/strain components are zero, as the transverse shear terms along the gauge length. The mixed moment resultant and curvature are zero in both the cases, since they are null at the xz symmetry plane, and they are assumed TW constant. By applying the costitutive relations proper of the homogeneous, isotropic plates, we derive for the unconstrained anticlastic curvature case i)

$$m_{\mathrm{x}} = m_{\mathrm{x}}^{*} \qquad m_{\mathrm{y}} = 0 \qquad \kappa_{\mathrm{x}} = k_{\mathrm{x}}^{*} \qquad \kappa_{\mathrm{y}} = -\nu k_{\mathrm{x}}^{*},$$

whereas for the cylindrical bending case ii) we have

$$m_{\mathrm{x}} = m_{\mathrm{x}}^{*} \qquad m_{\mathrm{y}} = \nu m_{\mathrm{x}}^{*} \qquad \kappa_{\mathrm{x}} = \left(1 - \nu^{2}\right) k_{\mathrm{x}}^{*} \qquad \kappa_{\mathrm{y}} = 0.$$

We then observe that the nonzero κ_{y} transverse curvature predicted by i) is inconsistent with the hypothesis of a full width line contact at the supports, whose cylindrical surface is transversely flat; the unconstrained anticlastic curvature confines the specimen contact interaction with the inner supports to a point in correspondence of the width midspan, whereas the outer supports touch the specimen at its edges only. Such a TW inhomogeneous loading condition induces contact actions which may effectively oppose the anticlastic curvature, which locally appears not "unconstrained" anymore.

On the other hand, a m_{y} moment resultant which is predicted according to cylindrical bending not to vanish at the specimen flanks is incompatible with the free surface boundary condition; continuity conditions requires in fact that a distributed moment external action is applied at the specimen flanks, which apparently is not the case.

The actual response of the specimen in terms of moment resultants and curvatures, as probed at its centroidal axis, is plotted in Fig. 5b in the case of bilateral support condition, i.e. $w = 0$ and $w = d$ at the outer and inner indenters, respectively, being d displacement of the inner, moving, support. The cylindrical bending solution ii) is observed at the supports, whereas a progressive transition to the unconstrained anticlastic curvature solution i) is observed while moving away from those supported areas. In particular, the central portion of the gauge length behaves consistently with i).

In Fig. 5c, the same quantities are reported in the actual case of unilateral contact at supports, i.e. the Signorini conditions[18] are

[18] Those conditions consist in turn in a no compenetration inequality 35, in a no tractive contact action inequality 36, and in the mutual local exclusion of nozero gap and nonzero contact force, 37.

imposed which consist in

$$g(y) \geq 0 \tag{35}$$

$$f(y) \geq 0 \tag{36}$$

$$g(y) \cdot f(y) = 0, \tag{37}$$

where $f(y)$ is the lineic contact force along the width, positive if compressive, and $g(y)$ is the gap between specimen and indenter, namely $g(y) = -w(y)$ and $g(y) = w(y) - d$ at the outer and inner supports, respectively.

According to this second model, supports are less effective in locally imposing a null secondary curvature, thus extending the validity of the unconstrained anticlastic curvature solution i) to most of the gauge length.

0.1.8 Final Notes.

A few sparse notes:

- If the unsymmetric laminate is composed by isotropic layers, a reference plane may be obtained for which the $\underline{\underline{B}}$ membrane-to-bending coupling matrix vanishes; a similar condition may not be verified in the presence of orthotropic layers.

- Thermally induced distortion is not self-compensated in an unsymmetric laminate even if the temperature is held constant through the thickness. Such fact, united to the unavoidable thermal cycles that occurs in manufacturing if not in operation, makes such configurations pretty undesirable.

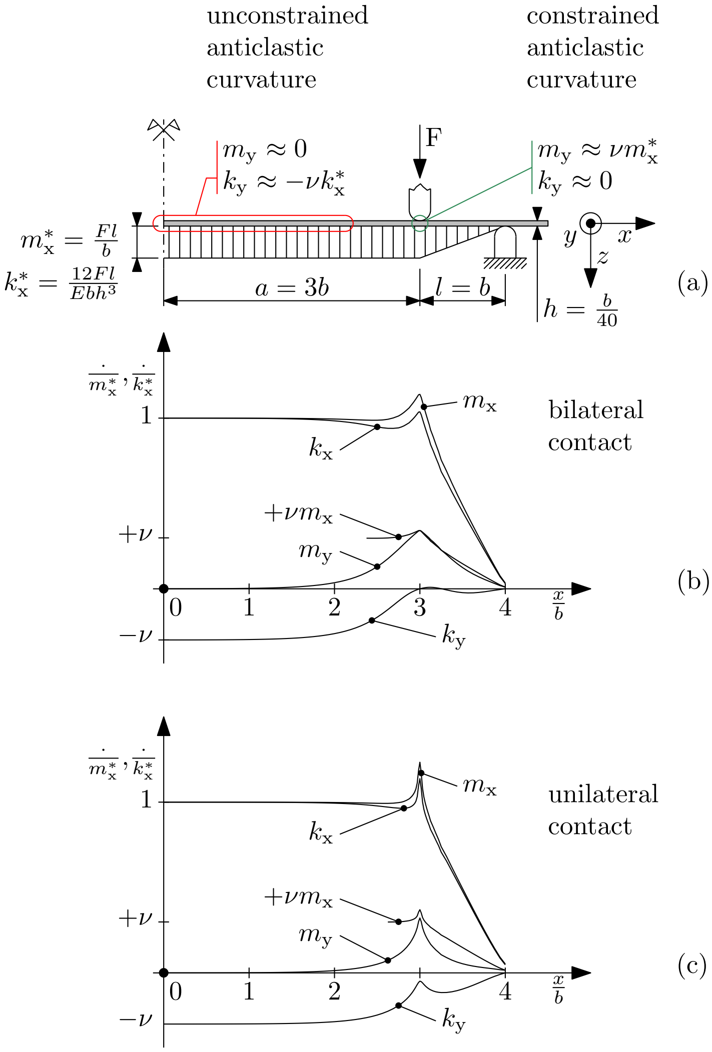

Figure 5: The *not-so-trivial* four point bending case, where b is the specimen out-of-sketch-plane width (we might call it *depth*). Moment fluxes and curvatures are sampled at the specimen midwidth, whereas they may vary while moving towards the flanks; the average value of m_x along the width must in fact coincide with m_x^* in correspondence with the load span.

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