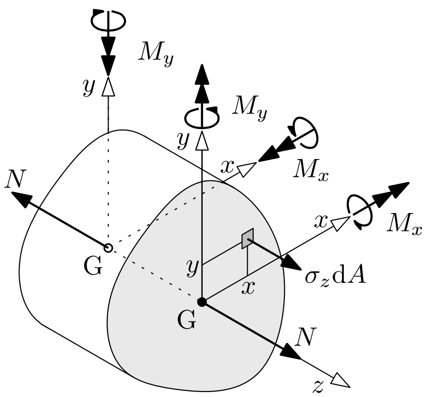
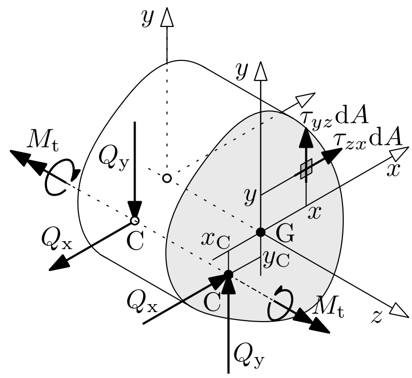
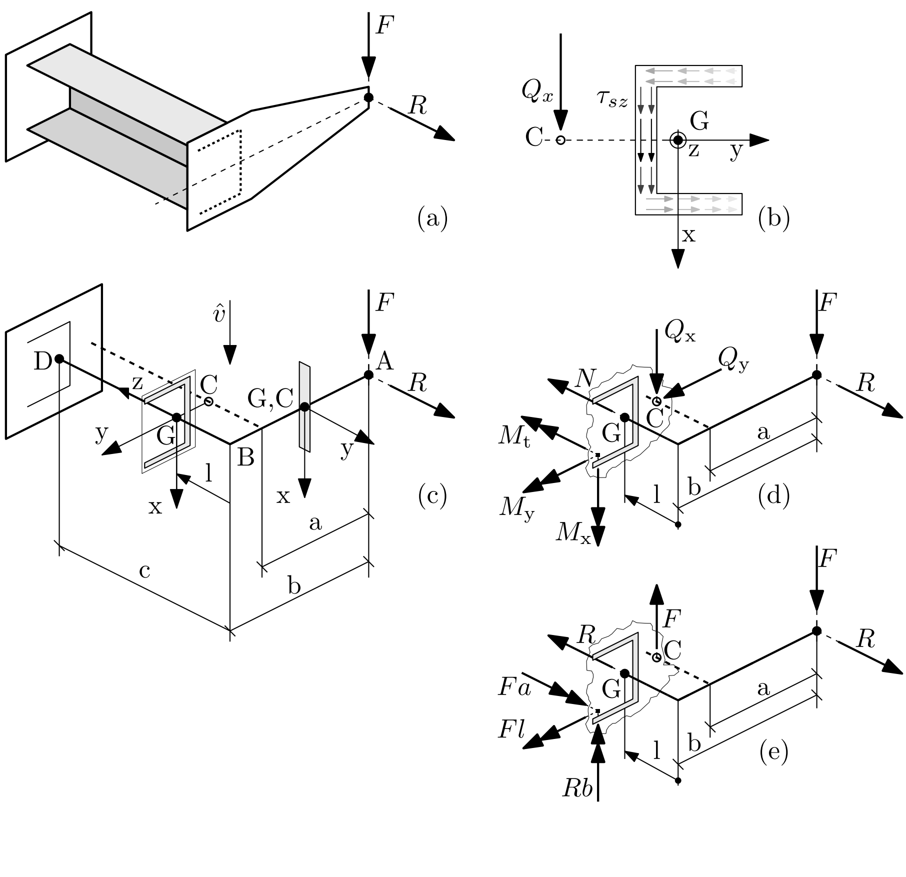




$$N = \int_{\mathcal{A}} \sigma_{zz} d\mathcal{A}$$
$$Q_y = \int_{\mathcal{A}} \tau_{yz} d\mathcal{A}$$
$$Q_x = \int_{\mathcal{A}} \tau_{zx} d\mathcal{A}$$

$$M_x \equiv M_{(G,x)} = \int_{\mathcal{A}} \sigma_z y d\mathcal{A}$$
$$M_y \equiv M_{(G,y)} = -\int_{\mathcal{A}} \sigma_z x d\mathcal{A}$$
$$M_t \equiv M_{(C,z)} = \int_{\mathcal{A}} \left[\tau_{yz}(x - x_C) - \tau_{zx}(y - y_C)\right] d\mathcal{A}$$



F
R
(a)
Q_x
τ_sz
G
C
z
y
x
(b)
v̂
F
D
z
C
G,C
A
R
y
G
B
y
l
x
x
a
c
b
(c)
F
Q_x
Q_y
N
R
M_t
G
C
a
M_y
l
b
(d)
M_x
F
F
R
C
R
Fa
G
a
Fl
l
b
(e)
Rb

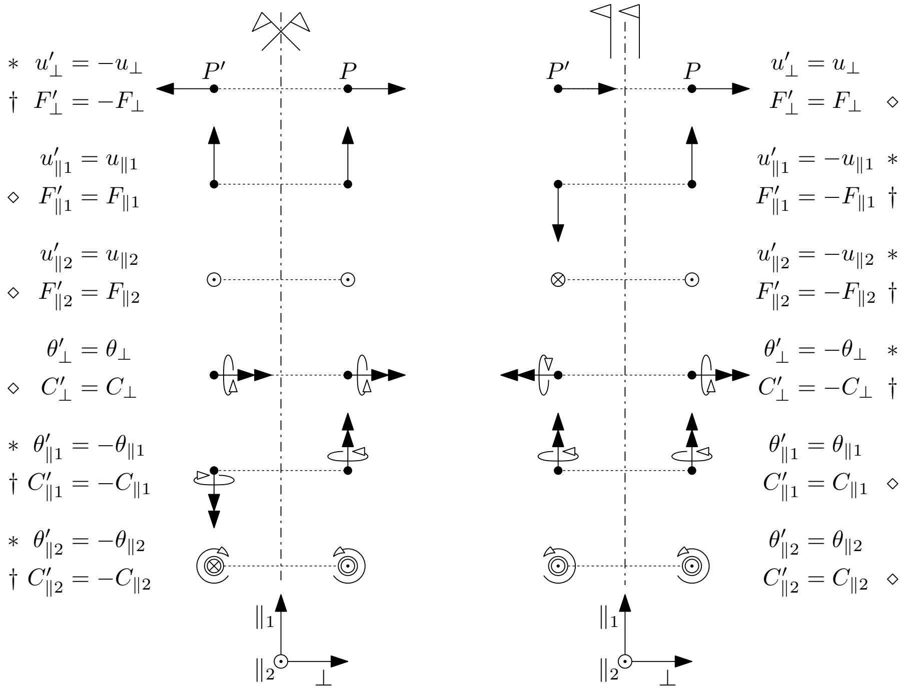
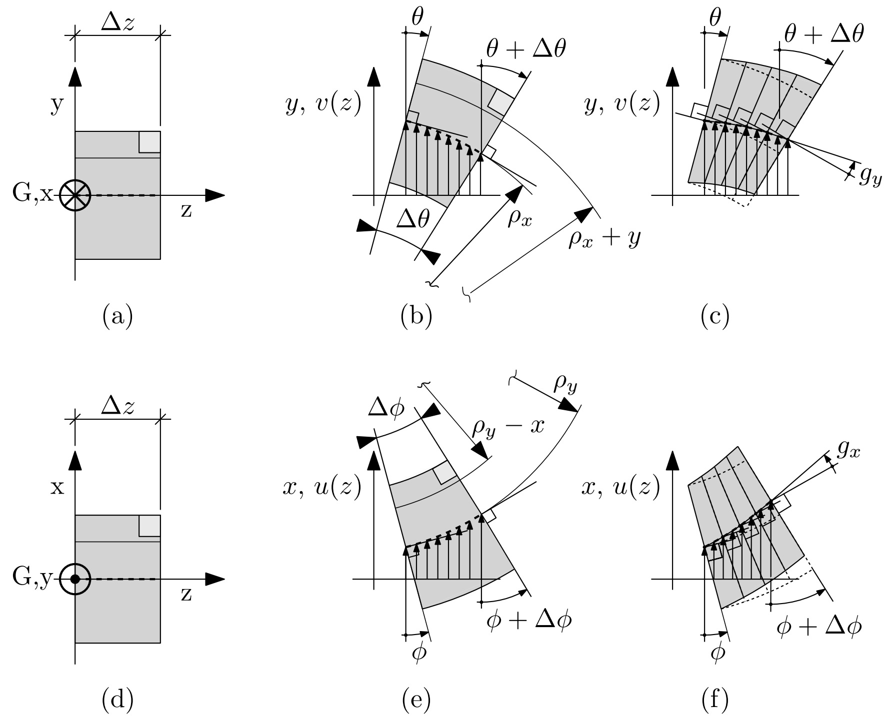


Figure 1: An overview of symmetrical and skew-symmetrical (generalized) loading and displacements.



$$\epsilon_z = a + bx + cy \tag{1}$$

$$\frac{d\theta}{dz} = \frac{1}{\rho_x}, \quad \frac{dv}{dz} = -\theta\ + [g_y]\,, \quad \frac{d^2v}{dz^2} = -\frac{1}{\rho_x} \tag{2}$$

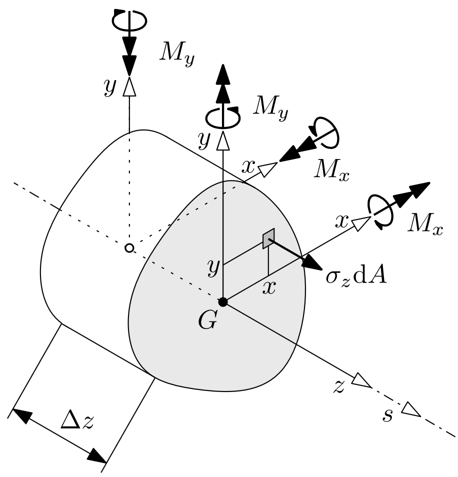
$$\frac{d\phi}{dz} = \frac{1}{\rho_y}, \quad \frac{du}{dz} = +\phi\ + [g_x]\,, \quad \frac{d^2u}{dz^2} = +\frac{1}{\rho_y} \tag{3}$$

$$\sigma_z = E_z\epsilon_z; \quad \epsilon_z = e - \frac{1}{\rho_y}x + \frac{1}{\rho_x}y \tag{4}$$

$$N = \iint_{\mathcal{A}} E_z\epsilon_z dA = \overline{EA}e \tag{5}$$

$$M_x = \iint_{\mathcal{A}} E_z\epsilon_z y dA = \overline{EJ}_{xx}\frac{1}{\rho_x} - \overline{EJ}_{xy}\frac{1}{\rho_y} \tag{6}$$

$$M_y = -\iint_{\mathcal{A}} E_z\epsilon_z x dA = -\overline{EJ}_{xy}\frac{1}{\rho_x} + \overline{EJ}_{yy}\frac{1}{\rho_y} \tag{7}$$



$$\overline{EA} = \iint_{\mathcal{A}} E_z(x,y)\; dA \tag{8}$$

$$\overline{EJ}_{xx} = \iint_{\mathcal{A}} E_z(x,y) yy\; dA \tag{9}$$

$$\overline{EJ}_{xy} = \iint_{\mathcal{A}} E_z(x,y) yx\; dA \tag{10}$$

$$\overline{EJ}_{yy} = \iint_{\mathcal{A}} E_z(x,y) xx\; dA \tag{11}$$

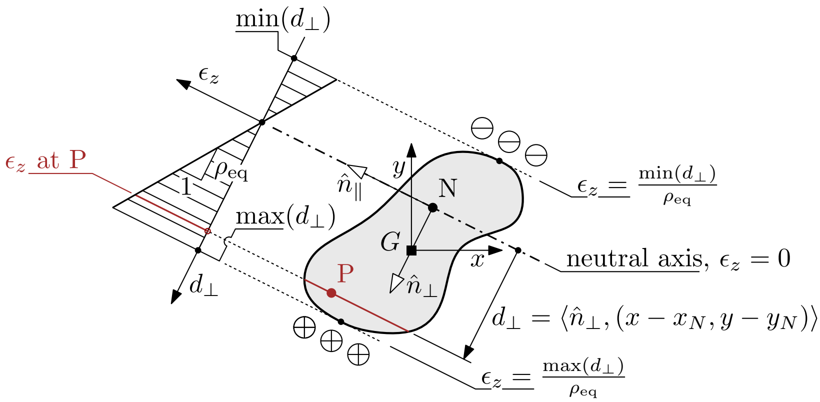
$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{EJ}_{xx} & -\overline{EJ}_{xy} \\ -\overline{EJ}_{xy} & \overline{EJ}_{yy} \end{bmatrix}}_{[\overline{EJ}]} \begin{bmatrix} \frac{1}{\rho_x} \\ \frac{1}{\rho_y} \end{bmatrix} \tag{12}$$

$$e = \frac{N}{\overline{EA}}. \tag{13}$$

$$\frac{1}{\rho_x} = \frac{M_x \overline{EJ}_{yy} + M_y \overline{EJ}_{xy}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{14}$$

$$\frac{1}{\rho_y} = \frac{M_x \overline{EJ}_{xy} + M_y \overline{EJ}_{xx}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{15}$$

$$\begin{bmatrix} \frac{1}{\rho_x} \\ \frac{1}{\rho_y} \end{bmatrix} = [\overline{EJ}]^{-1} \begin{bmatrix} M_x \\ M_y \end{bmatrix} \tag{16}$$



$$[\overline{EJ}]^{-1} = \frac{1}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \begin{bmatrix} \overline{EJ}_{yy} & \overline{EJ}_{xy} \\ \overline{EJ}_{xy} & \overline{EJ}_{xx} \end{bmatrix}$$

$$\frac{1}{\rho_{\mathrm{eq}}} = \sqrt{\frac{1}{\rho_x^2} + \frac{1}{\rho_y^2}}, \tag{17}$$

$$\sigma_z = E_z \epsilon_z; \quad \epsilon_z = \begin{bmatrix} y & -x \end{bmatrix} [\overline{EJ}]^{-1} \begin{bmatrix} M_x \\ M_y \end{bmatrix} + \frac{1}{\overline{EA}} N \tag{18}$$

$$(x_N, y_N) \equiv e \rho_{\mathrm{eq}}^2 \left(\frac{1}{\rho_y}, -\frac{1}{\rho_x} \right);$$

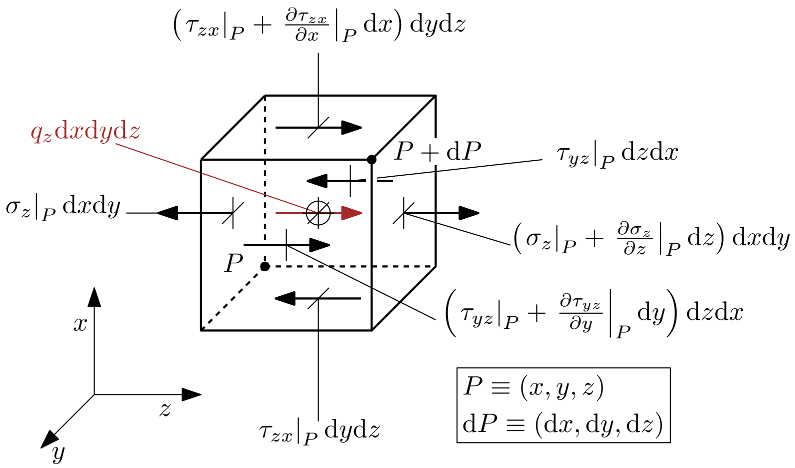
$$\hat{n}_\parallel = \rho_{\mathrm{eq}} \left(\frac{1}{\rho_x}, \frac{1}{\rho_y} \right),$$

$$\hat{n}_\perp = \rho_{\mathrm{eq}} \left(-\frac{1}{\rho_y}, \frac{1}{\rho_x} \right),$$

$$\epsilon_z = \frac{1}{\rho_{\mathrm{eq}}} \underbrace{\langle \hat{n}_\perp, (x - x_N, y - y_N) \rangle}_{d_\perp} = \frac{1}{\rho_{\mathrm{eq}}} d_\perp \tag{19}$$

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \zeta \hat{n}_\parallel = \lambda \begin{bmatrix} \frac{1}{\rho_x} \\ \frac{1}{\rho_y} \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{EJ}_{xx} & -\overline{EJ}_{xy} \\ -\overline{EJ}_{xy} & \overline{EJ}_{yy} \end{bmatrix}}_{[\overline{EJ}]} \begin{bmatrix} \frac{1}{\rho_x} \\ \frac{1}{\rho_y} \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{\rho_x} \\ \frac{1}{\rho_y} \end{bmatrix} \tag{21}$$

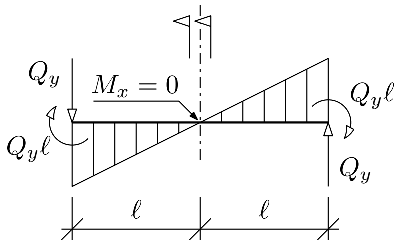


$$Q_y = \frac{dM_x}{dz}, \quad Q_x = -\frac{dM_y}{dz}, \tag{22}$$

$$\frac{d\sigma_z}{dz} = E_z \begin{bmatrix} y & -x \end{bmatrix} \left[\overline{EJ}\right]^{-1} \begin{bmatrix} Q_y \\ -Q_x \end{bmatrix} \tag{23}$$

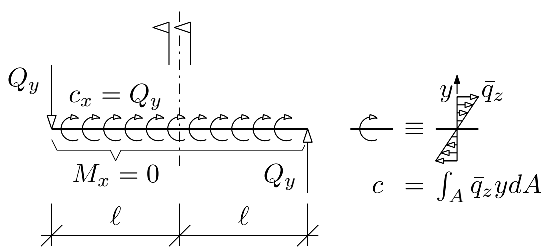
$$\frac{d\sigma_z}{dz} = E_z \begin{bmatrix} x & y \end{bmatrix} \frac{[\overline{EJ}]}{\det\left([\overline{EJ}]\right)} \begin{bmatrix} Q_x \\ Q_y \end{bmatrix}. \tag{24}$$

$$\frac{d\tau_{zx}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + q_z = 0 \tag{25}$$



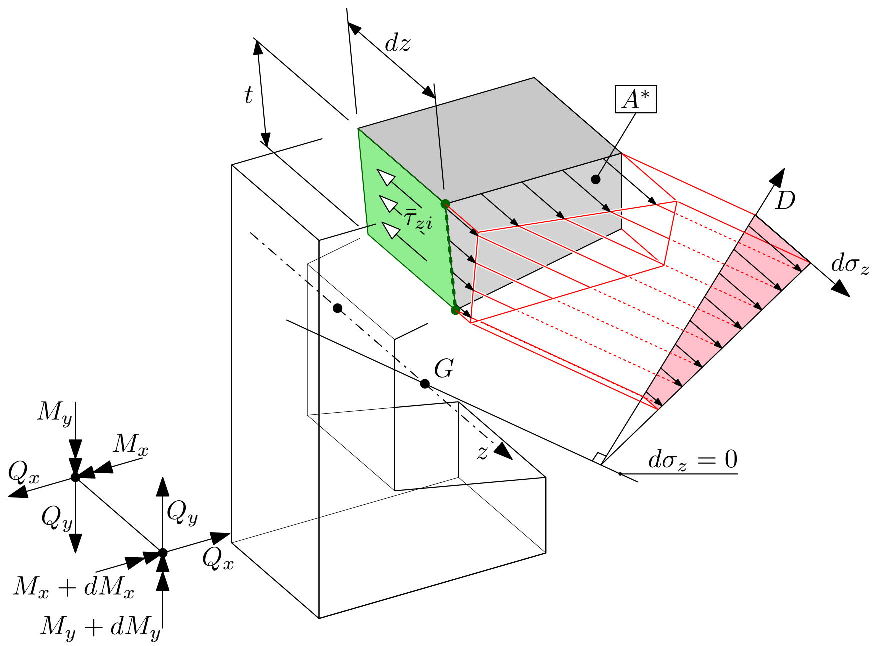
Q_y
$M_x = 0$
$Q_y\ell$
$Q_y\ell$
Q_y
ℓ
ℓ

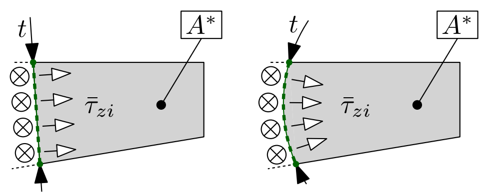
(a)



Q_y
$c_x = Q_y$
$M_x = 0$
Q_y
ℓ
ℓ
y
$\bar{q}_z$
$\equiv$
$c \; = \int_A \bar{q}_z y dA$

(b)





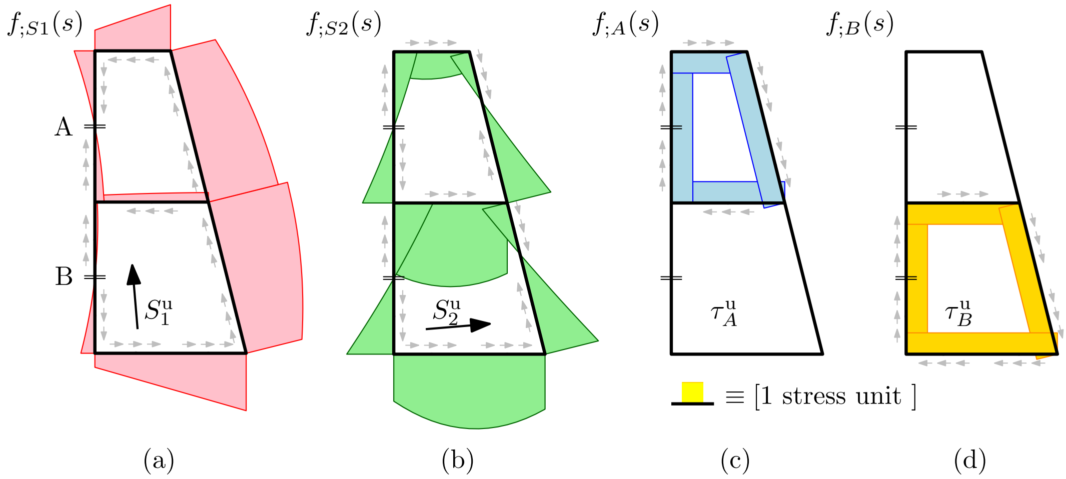
$$\bar{\tau}_{zi}t = \int_{A^*} \frac{d\sigma_z}{dz} dA, \tag{26}$$

$$\bar{\tau}_{zi} = \frac{1}{t}\int_t \tau_{zi} dr \tag{27}$$

$$\bar{\tau}_{zi}t = \int_{A^*} \left(\frac{yQ_y}{J_{xx}} + \frac{xQ_x}{J_{yy}} \right) dA = \frac{\bar{y}^* A^*}{J_{xx}} Q_y + \frac{\bar{x}^* A^*}{J_{yy}} Q_x, \tag{28}$$

$$\bar{\tau}_{zi}t = q_{zi} = \int_0^s \int_{-t/2}^{t/2} \frac{d\sigma_z}{dz} dr d\varsigma \approx \int_0^s \left. \frac{d\sigma_z}{dz} \right|_{r=0} t d\varsigma. \tag{29}$$

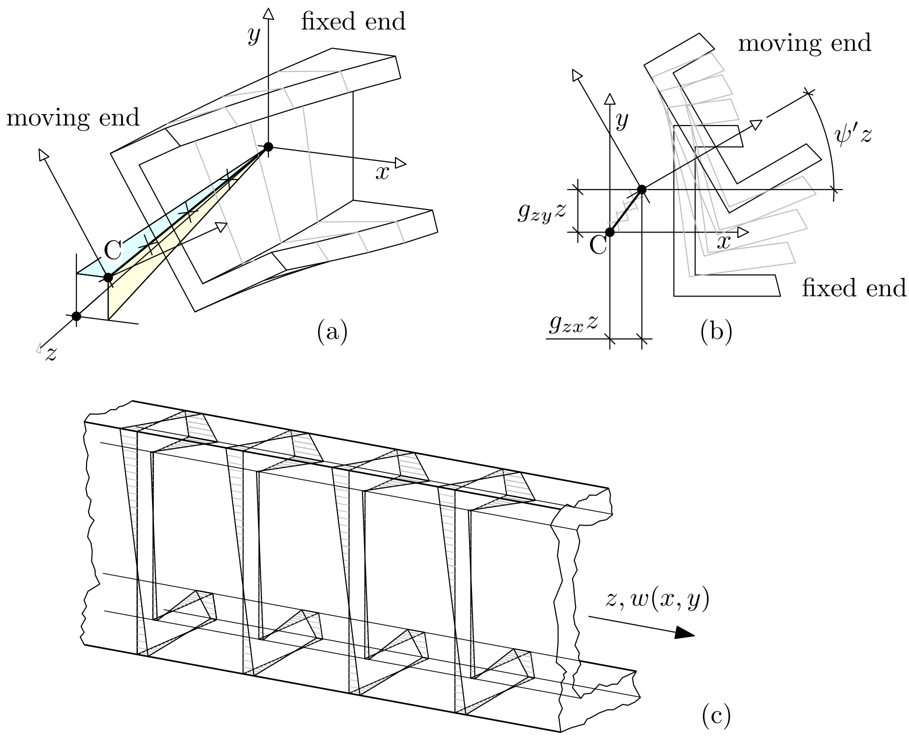
$$\bar{\tau}_{zi}(s)t(s) = q(s) = \int_a^s \frac{d\sigma_z}{dz} t d\varsigma + \underbrace{\bar{\tau}_{zi}(a)t(a)}_{q_A}. \tag{30}$$



$$\tau(s) = \frac{Q_1}{\mathcal{A}} f_{;S1}(s) + \frac{Q_2}{\mathcal{A}} f_{;S2}(s) + \tau_A f_{;A}(s) + \tau_B f_{;B}(s) \tag{31}$$

$$\Delta U = \int_s \frac{\tau^2}{2G_{sz}} t \Delta z ds \tag{32}$$

$$\frac{\partial \Delta U}{\partial \bar{\tau}_i} = \bar{\delta}_i t \Delta z \tag{33}$$



$$K_t = \frac{4A^2}{\oint \frac{1}{t} dl} \tag{34}$$

$$\tau_{\max} = \frac{M_t}{2t_{\min} A} \tag{35}$$

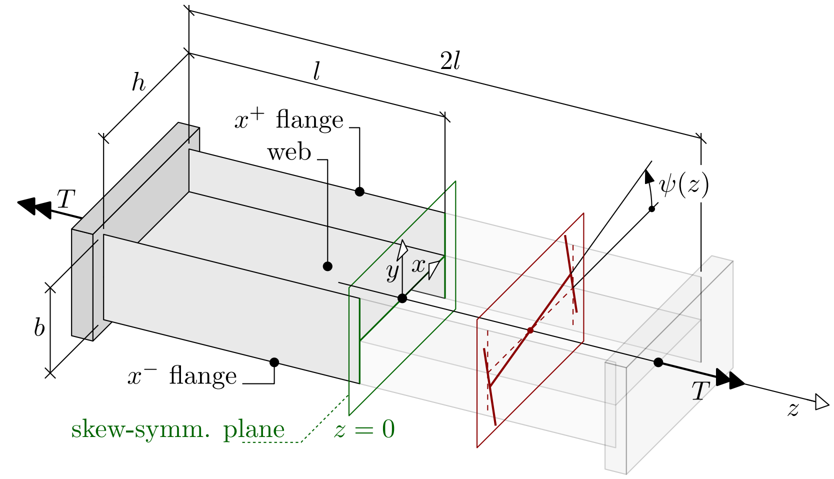
.

$$K_T \approx \frac{1}{3} \int_0^l t^3(s) ds \tag{36}$$

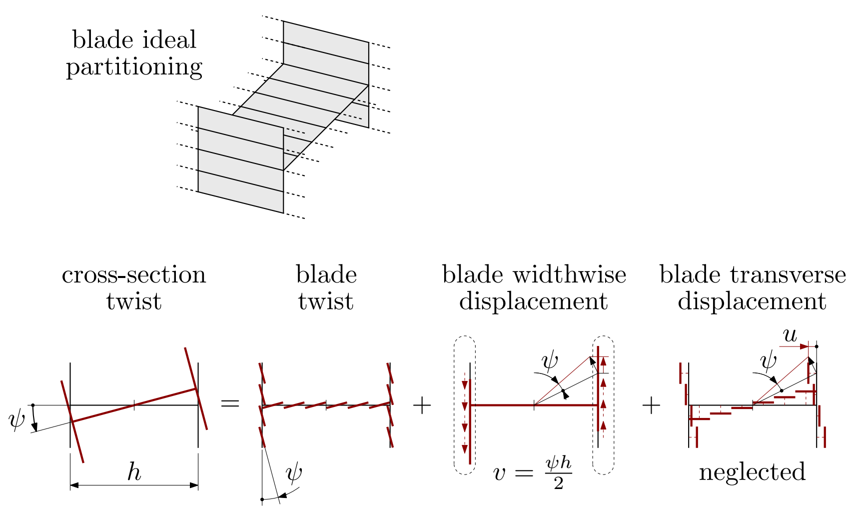
$$K_T \approx \frac{1}{3} \sum_i l_i t_i^3 \tag{37}$$

$$\tau_{\max} = \frac{M_t t_{\max}}{K_T} \tag{38}$$

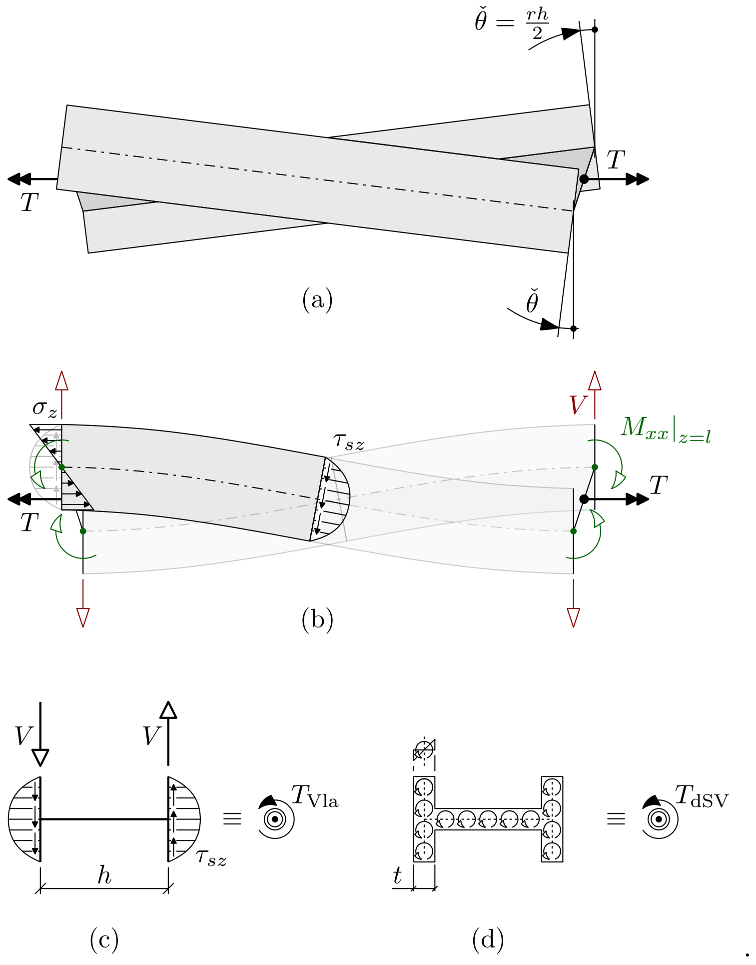
$$V = \frac{dM_x}{dz}.$$



2l
l
h
x^+ flange
web
T
$\psi(z)$
y
x
b
x^- flange
T
z
skew-symm. plane
$z=0$



blade ideal partitioning
cross-section twist
blade twist
blade widthwise displacement
blade transverse displacement
ψ
h
=
+
$v=\frac{\psi h}{2}$
u
neglected



$\breve{\theta} = \frac{rh}{2}$
T
T
(a)
$\breve{\theta}$
σ_z
V
$M_{xx}|_{z=l}$
τ_{sz}
T
T
(b)
V
V
τ_{sz}
h
≡
T_{Vla}
(c)
t
≡
T_{dSV}
(d)

$$T_{\text{Vla}} = hV = h\frac{dM_x}{dz};$$

$$M_x = \frac{EJ_{xx}}{\rho_x} = -EJ_{xx}\frac{d^2v}{dz^2}, \quad EJ_{xx} = \frac{Eb^3t}{12}$$

$$T_{\text{Vla}} = -h\overline{EJ}_{xx}\frac{d^3v}{dz^3},$$

$$v = \frac{h}{2}\psi,$$

$$T_{\text{Vla}} = -\frac{h^2}{2}EJ_{xx}\frac{d^3\psi}{dz^3} = -EC_w\frac{d^3\psi}{dz^3}$$

$$EC_w = EJ_{xx}\frac{h^2}{2} = \frac{Eb^3th^2}{24}$$

$$T_{\text{dSV}} = GK_t\frac{d\psi}{dz},$$

$$d = \sqrt{\frac{EC_w}{GK_t}}, \quad EC_w = d^2GK_t$$

$$0 = \frac{dT_{\text{dSV}}}{dz} + \frac{dT_{\text{Vla}}}{dz} = -EC_w\frac{d^4\psi}{dz^4} + GK_t\frac{d^2\psi}{dz^2}$$

$$0 = -d^2\frac{d^4\psi}{dz^4} + \frac{d^2\psi}{dz^2}$$

$$\psi(z) = C_1\sinh\frac{z}{d} + C_2\cosh\frac{z}{d} + C_3\frac{z}{d} + C_4 \tag{39}$$

$$B = M_x \cdot h$$

$$B = -EC_w\frac{d^2\psi}{dz^2};$$

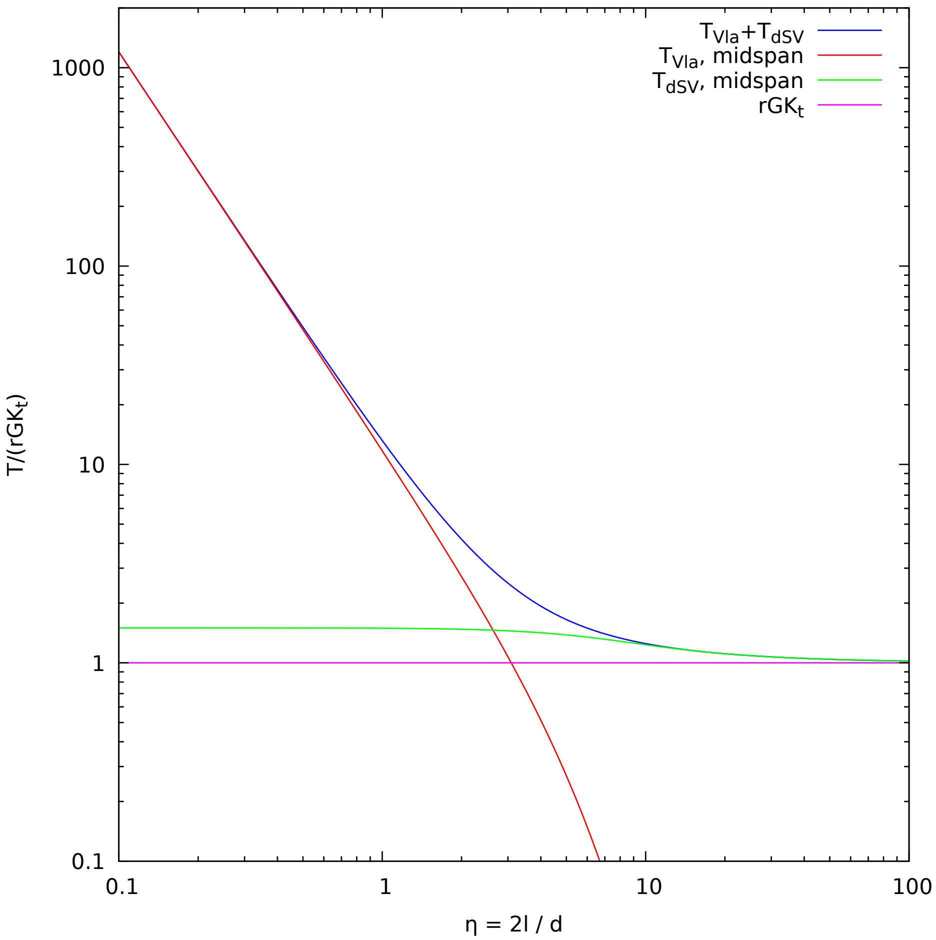
$$\{C_1, C_3\} = rl \cdot \frac{\left\{-1, \cosh\left(\frac{l}{d}\right)\right\}}{\frac{l}{d}\cosh\left(\frac{l}{d}\right) - \sinh\left(\frac{l}{d}\right)} \tag{40}$$

$$\psi|_{z=l} = rl, \quad \left.\frac{d\psi}{dz}\right|_{z=l} = 0.$$

$$\frac{T_{\text{Vla}} + T_{\text{dSV}}}{GK_tr} = \frac{\eta}{\eta - 2\tanh\left(\frac{\eta}{2}\right)} = S(\eta). \tag{41}$$

$$\eta_{i+1} = \eta_i + \frac{\bar{S} - S_i}{S_i'}, \quad S_i = S(\eta_i), \quad S_i' = \left.\frac{\partial S}{\partial\eta}\right|_{\eta=\eta_i}$$

$$\bar{d} = \frac{2\bar{l}}{\eta^*}.$$



T/(rGK$_t$)
1000
100
10
1
0.1
0.1
1
10
100
η = 2l / d
T$_{Vla}$+T$_{dSV}$
T$_{Vla}$, midspan
T$_{dSV}$, midspan
rGK$_t$

$$q_i = \frac{\partial U}{\partial Q_i}$$

$$\frac{dU}{dl} = \frac{1}{2}\begin{pmatrix} N \\ M_\mathrm{x} \\ M_\mathrm{y} \\ Q_\mathrm{x} \\ Q_\mathrm{y} \\ M_\mathrm{t} \end{pmatrix}^\top \begin{pmatrix} a_{1,1} & g_{1,2} & g_{1,3} & i_{1,4} & i_{1,5} & i_{1,6} \\ 0 & b_{2,2} & e_{2,3} & i_{2,4} & i_{2,5} & i_{2,6} \\ 0 & 0 & b_{3,3} & i_{3,4} & i_{3,5} & i_{3,6} \\ 0 & 0 & 0 & c_{4,4} & f_{4,5} & h_{4,6} \\ 0 & 0 & 0 & 0 & c_{5,5} & h_{5,6} \\ 0 & 0 & 0 & 0 & 0 & d_{6,6} \end{pmatrix}_\mathrm{Sym} \begin{pmatrix} N \\ M_\mathrm{x} \\ M_\mathrm{y} \\ Q_\mathrm{x} \\ Q_\mathrm{y} \\ M_\mathrm{t} \end{pmatrix}, \tag{42}$$

$$a_{1,1} = \frac{1}{EA} \qquad \{b_{2,2}, b_{3,3}, e_{2,3}\} = \frac{\{J_\mathrm{yy}, J_\mathrm{xx}, 2J_\mathrm{xy}\}}{E\left(J_\mathrm{xx}J_\mathrm{yy} - J_\mathrm{xy}^2\right)}$$

$$d_{6,6} = \frac{1}{GK_t} \qquad \{c_{4,4}, c_{5,5}, f_{4,5}\} = \frac{\{\chi_\mathrm{x}, \chi_\mathrm{y}, \chi_\mathrm{xy}\},}{GA}$$

$$\begin{pmatrix} e \\ \frac{1}{\rho_\mathrm{x}} \\ \frac{1}{\rho_\mathrm{y}} \\ g_\mathrm{x} \\ g_\mathrm{y} \\ \psi' \end{pmatrix} = \begin{pmatrix} a_{1,1} & g_{1,2} & g_{1,3} & i_{1,4} & i_{1,5} & i_{1,6} \\ 0 & b_{2,2} & e_{2,3} & i_{2,4} & i_{2,5} & i_{2,6} \\ 0 & 0 & b_{3,3} & i_{3,4} & i_{3,5} & i_{3,6} \\ 0 & 0 & 0 & c_{4,4} & f_{4,5} & h_{4,6} \\ 0 & 0 & 0 & 0 & c_{5,5} & h_{5,6} \\ 0 & 0 & 0 & 0 & 0 & d_{6,6} \end{pmatrix}_{\underline{\mathrm{Sym}}} \begin{pmatrix} N \\ M_\mathrm{x} \\ M_\mathrm{y} \\ Q_\mathrm{x} \\ Q_\mathrm{y} \\ M_\mathrm{t} \end{pmatrix}, \tag{43}$$

$$
\begin{aligned}
u_P &= u + z\,(1+\check{\epsilon}_z)\,\frac{\cos\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}}\sin\phi \\
v_P &= v - z\,(1+\check{\epsilon}_z)\,\frac{\cos\phi}{\sqrt{1-\sin^2\phi\sin^2\theta}}\sin\theta \\
w_P &= w + z\left((1+\check{\epsilon}_z)\,\frac{\cos\phi\cos\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}} - 1\right),
\end{aligned}
$$

$$
\begin{aligned}
\check{\epsilon}_z(z) &= \frac{1}{z}\int_0^z \epsilon_z d\varsigma \\
&= \frac{1}{z}\int_0^z -\frac{\nu}{1-\nu}\left(\epsilon_x+\epsilon_y\right) d\varsigma,
\end{aligned}
$$

$$u_P = u + z\phi \tag{44}$$

$$v_P = v - z\theta \tag{45}$$

$$w_P = w. \tag{46}$$

$$\frac{\partial w}{\partial x} = g_{zx} - \phi \tag{47}$$

$$\frac{\partial w}{\partial y} = g_{yz} + \theta \tag{48}$$

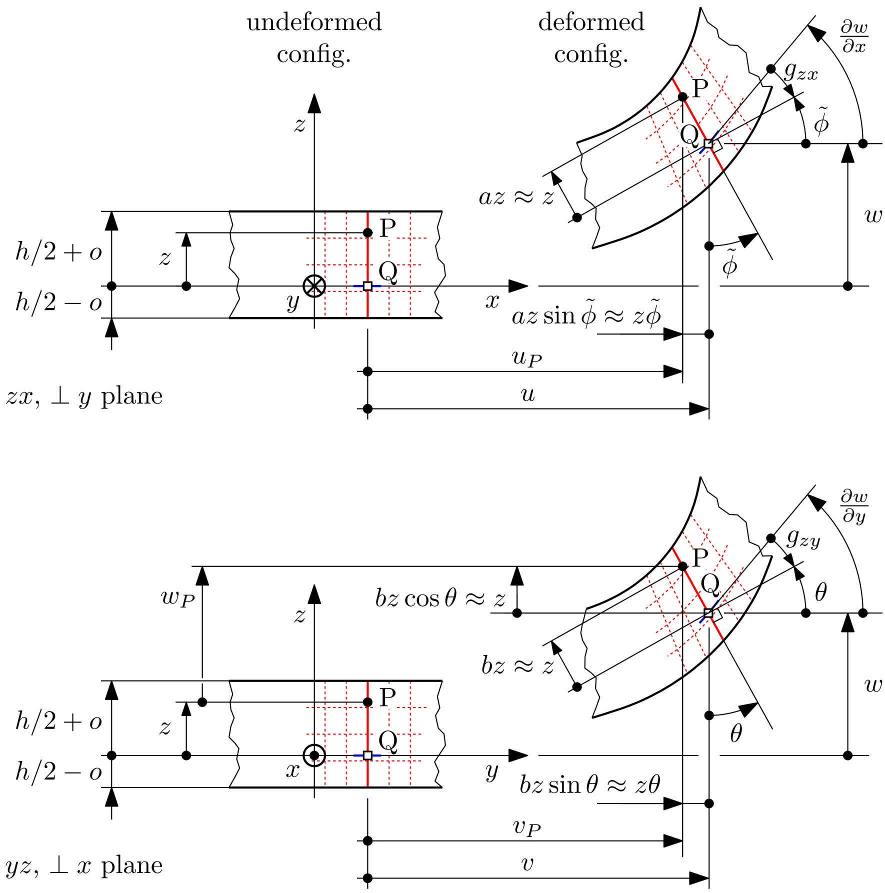
$$\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \tag{49}$$

$$\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial \theta}{\partial y} \tag{50}$$

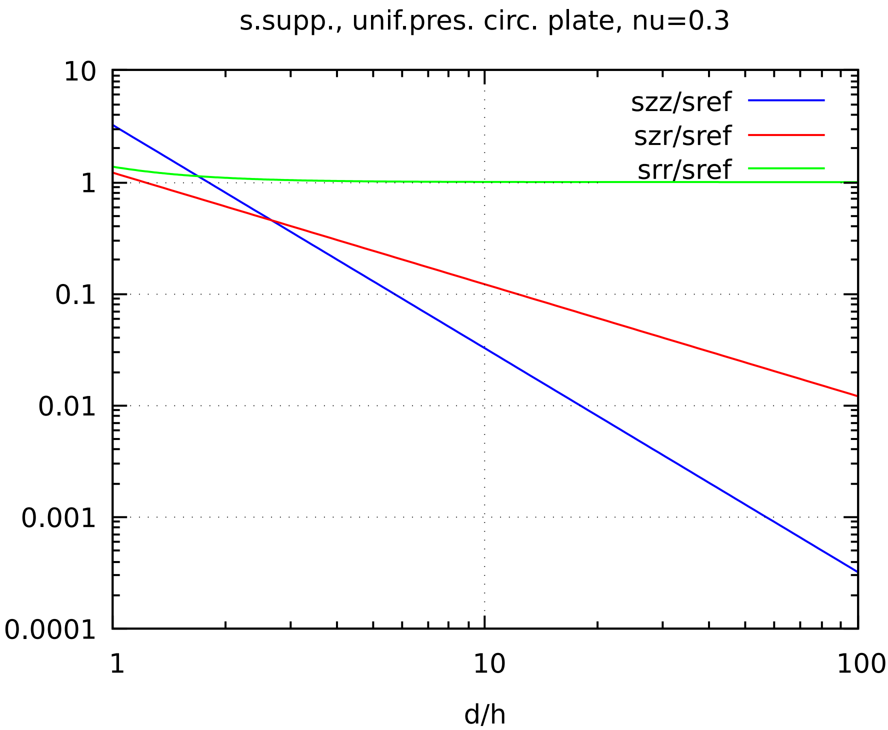
$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{51}$$

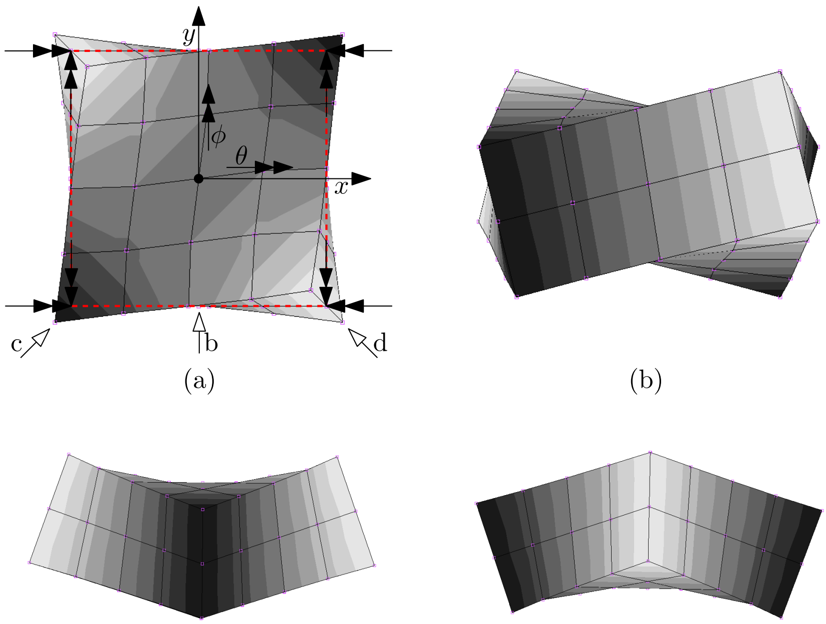
$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x}\right) \tag{52}$$

$$\underline{\mathrm{e}} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} e_x \\ e_y \\ g_{xy} \end{bmatrix} \equiv \underline{\epsilon}_{\mathrm{Q}} \tag{53}$$



$$\{a, b\} = \frac{(1+\check{\epsilon})}{\sqrt{1-\sin^2\phi\sin^2\theta}}\{\cos\theta, \cos\phi\} \approx 1$$


s.supp., unif.pres. circ. plate, nu=0.3
10
1
0.1
0.01
0.001
0.0001
1
10
100
d/h
szz/sref
szr/sref
srr/sref



(a) (b) (c) (d)

$$\underline{\kappa} = \begin{bmatrix} +\frac{\partial \phi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x} \end{bmatrix} = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{54}$$

$$\underline{\epsilon}_{\,\mathrm{P}} \equiv \underline{\epsilon} = \underline{\mathrm{e}} + z\,\underline{\kappa}\,. \tag{55}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underline{\sigma} = \underline{\underline{\mathrm{D}}}\,\underline{\epsilon} = \underline{\underline{\mathrm{D}}}\,\underline{\mathrm{e}} + z\,\underline{\underline{\mathrm{D}}}\,\underline{\kappa}\,, \tag{56}$$

$$\underline{\underline{\mathrm{D}}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \tag{57}$$

$$\epsilon_z = -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right). \tag{58}$$

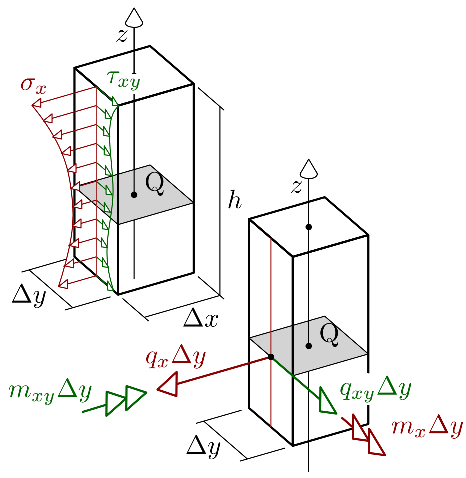


Figure 2: XXX

$$\underline{q} = \begin{bmatrix} q_x \\ q_y \\ q_{xy} \end{bmatrix} = \int_h \underline{\sigma}\, dz$$

$$= \underbrace{\int_h \underline{\underline{D}}\, dz}_{\underline{\underline{a}}}\ \underline{e} + \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{b}}}\ \underline{\kappa} \tag{59}$$

$$\underline{q}_z = \begin{bmatrix} q_{xz} \\ q_{yz} \end{bmatrix} \qquad q_{xz} = \int_h \tau_{zx} dz \qquad q_{yz} = \int_h \tau_{yz} dz. \tag{60}$$

$$\underline{m} = \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} = \int_h \underline{\sigma}\, z dz$$

$$= \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{b}} \equiv \underline{\underline{b}}^{\mathrm{T}}}\ \underline{e} + \underbrace{\int_h \underline{\underline{D}}\, z^2 dz}_{\underline{\underline{c}}}\ \underline{\kappa}\,. \tag{61}$$

$$\begin{bmatrix} \underline{q} \\ \underline{m} \end{bmatrix} = \begin{bmatrix} \underline{\underline{a}} & \underline{\underline{b}} \\ \underline{\underline{b}}^{\mathrm{T}} & \underline{\underline{c}} \end{bmatrix} \begin{bmatrix} \underline{e} \\ \underline{\kappa} \end{bmatrix} \tag{62}$$

$$\upsilon^{\dagger} = \frac{1}{2} \begin{bmatrix} \underline{q} \\ \underline{m} \end{bmatrix}^{\top} \begin{bmatrix} \underline{e} \\ \underline{\kappa} \end{bmatrix} \tag{63}$$

$$= \frac{1}{2} \begin{bmatrix} \underline{e} \\ \underline{\kappa} \end{bmatrix}^{\top} \begin{bmatrix} \underline{\underline{a}} & \underline{\underline{b}} \\ \underline{\underline{b}}^{\mathrm{T}} & \underline{\underline{c}} \end{bmatrix} \begin{bmatrix} \underline{e} \\ \underline{\kappa} \end{bmatrix}. \tag{64}$$

$$\underline{\underline{a}} = h\,\underline{\underline{D}} \qquad \underline{\underline{b}} = \underline{\underline{0}} \qquad \underline{\underline{c}} = \frac{h^3}{12}\,\underline{\underline{D}},$$

$$\underline{g}_{\,z} = \begin{bmatrix} g_{yz} \\ g_{zx} \end{bmatrix}$$

$$\underline{q}_{\,z} = \begin{bmatrix} q_{xz} \\ q_{yz} \end{bmatrix}$$

$$\upsilon^{\ddagger} = \frac{1}{2}\,\underline{g}_{\,z}^{\top}\,\underline{q}_{\,z} = \frac{1}{2} g_{xz} q_{xz} + \frac{1}{2} g_{yz} q_{yz}. \tag{65}$$

$$\upsilon^{\ddagger} = \frac{1}{2}\,\underline{g}_{\,z}^{\top} \underbrace{\left[\chi \left(\frac{1}{h} \int_h \underline{\underline{G}}^{\,-1} dz \right)^{-1} \right]}_{\underline{\underline{\Gamma}}} \underline{g}_{\,z} \tag{66}$$

$$\underline{\underline{G}} = \frac{E}{2\,(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

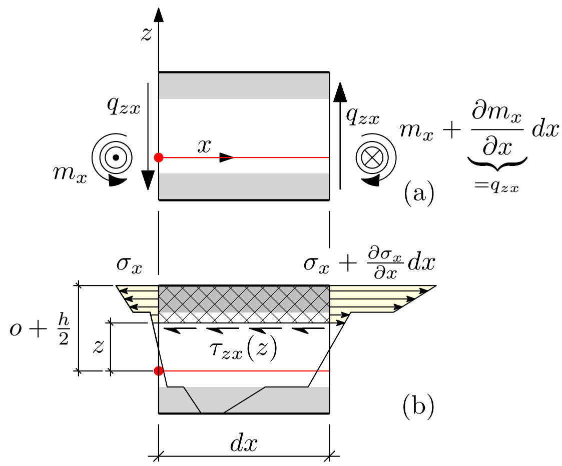
$$\underline{q}_{\,z} = \underline{\underline{\Gamma}}\,\underline{g}_{\,z}. \tag{67}$$

$$\tau_{zx}(z) = -\int_{-\frac{h}{2}+o}^{z} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} dz = \int_{z}^{+o+\frac{h}{2}} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} dz \tag{68}$$

$$\tau_{yz}(z) = -\int_{-\frac{h}{2}+o}^{z} \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} dz = \int_{z}^{+o+\frac{h}{2}} \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} dz. \tag{69}$$

$$\begin{bmatrix} \underline{q} \\ \underline{m} \\ \underline{q}_{\,z} \end{bmatrix} = \begin{bmatrix} \underline{\underline{a}} & \underline{\underline{b}} & \underline{\underline{0}} \\ \underline{\underline{b}}^{\mathrm{T}} & \underline{\underline{c}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\Gamma}} \end{bmatrix} \begin{bmatrix} \underline{e} \\ \underline{\kappa} \\ \underline{\gamma}_{\,z} \end{bmatrix} \tag{70}$$

$$\underline{\underline{D}}_{\,123} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{71}$$



$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \underline{\underline{T}}_1 \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \underline{\underline{T}}_2 \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{72}$$

$$\underline{\underline{T}}_1 = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \tag{73}$$

$$\underline{\underline{T}}_2 = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \tag{74}$$

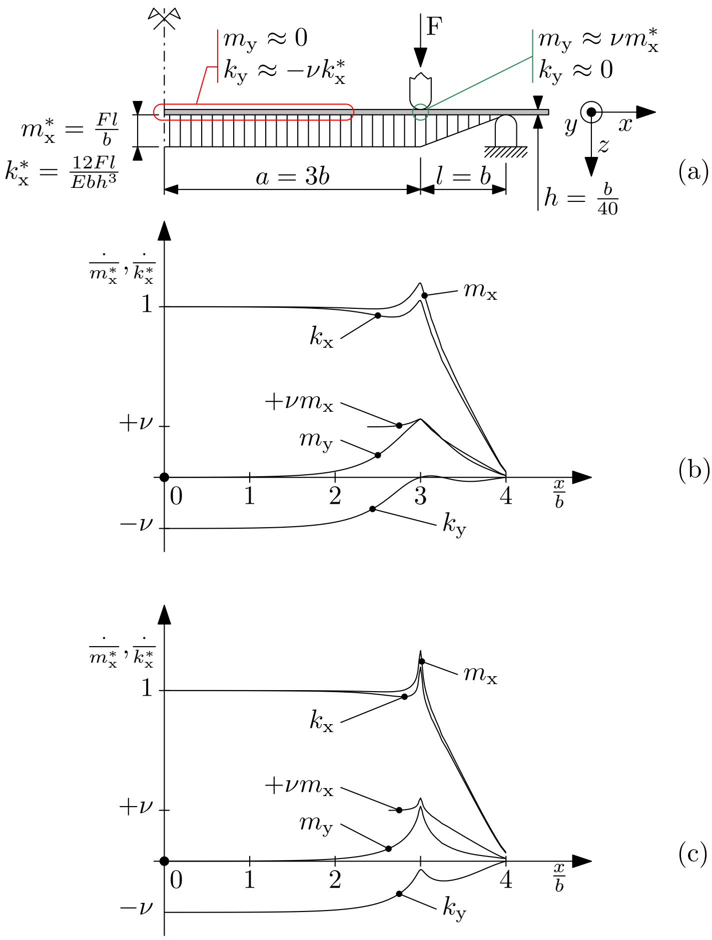
$$m = \cos(\alpha) \qquad n = \sin(\alpha) \tag{75}$$

$$\underline{\underline{T}}_1^{-1}(+\alpha) = \underline{\underline{T}}_1(-\alpha) \qquad \underline{\underline{T}}_2^{-1}(+\alpha) = \underline{\underline{T}}_2(-\alpha) \tag{76}$$

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} \qquad \underline{\underline{D}} \equiv \underline{\underline{D}}_{xyz} = \underline{\underline{T}}_1^{-1}\,\underline{\underline{D}}_{123}\,\underline{\underline{T}}_2 \tag{77}$$

$$\underline{\underline{G}} = \begin{bmatrix} n^2 G_{z1} + m^2 G_{2z} & mnG_{z1} - mnG_{2z} \\ mnG_{z1} - mnG_{2z} & m^2 G_{z1} + n^2 G_{2z} \end{bmatrix}.$$

$$k_{\mathrm{x}}^* = \frac{12Fl}{Ebh^3} \tag{78}$$



$$m_x = m_x^* \qquad m_y = 0 \qquad \kappa_x = k_x^* \qquad \kappa_y = -\nu k_x^*,$$

$$m_x = m_x^* \qquad m_y = \nu m_x^* \qquad \kappa_x = \left(1 - \nu^2\right) k_x^* \qquad \kappa_y = 0.$$

$$g(y) \geq 0 \tag{79}$$

$$f(y) \geq 0 \tag{80}$$

$$g(y) \cdot f(y) = 0, \tag{81}$$

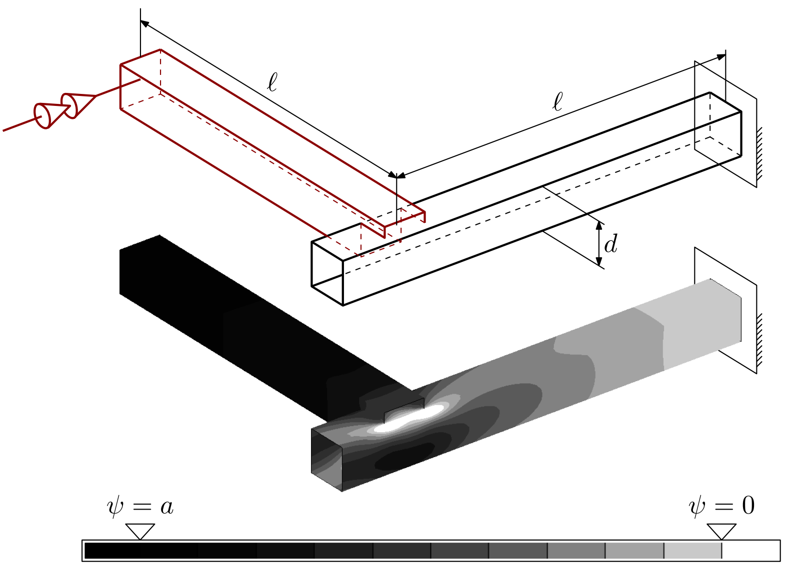


Figure 3: FIXME

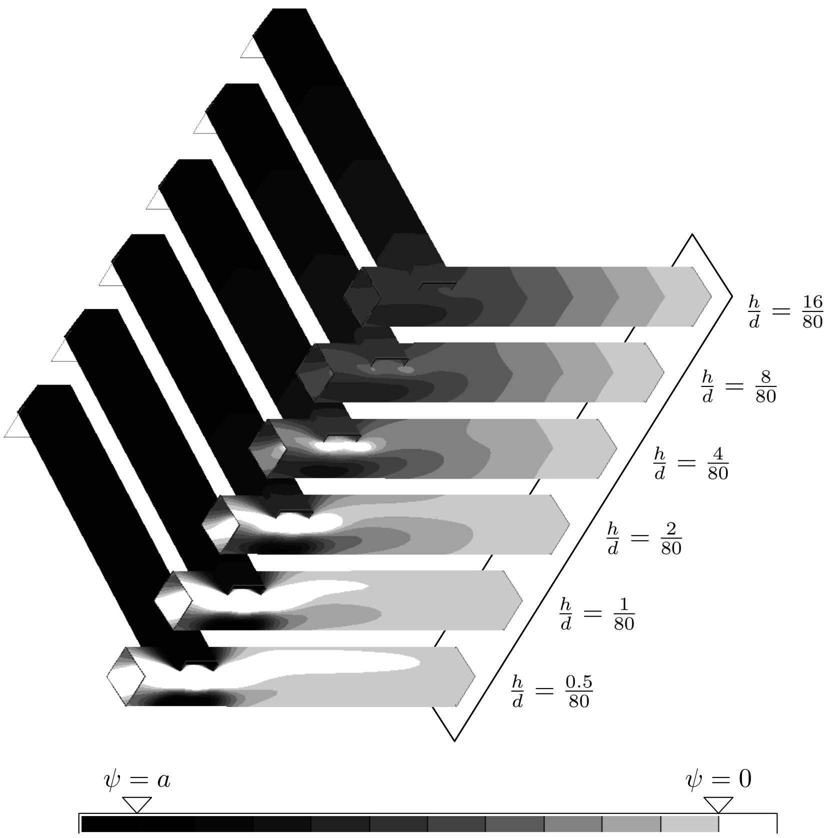


Figure 4: FIXME

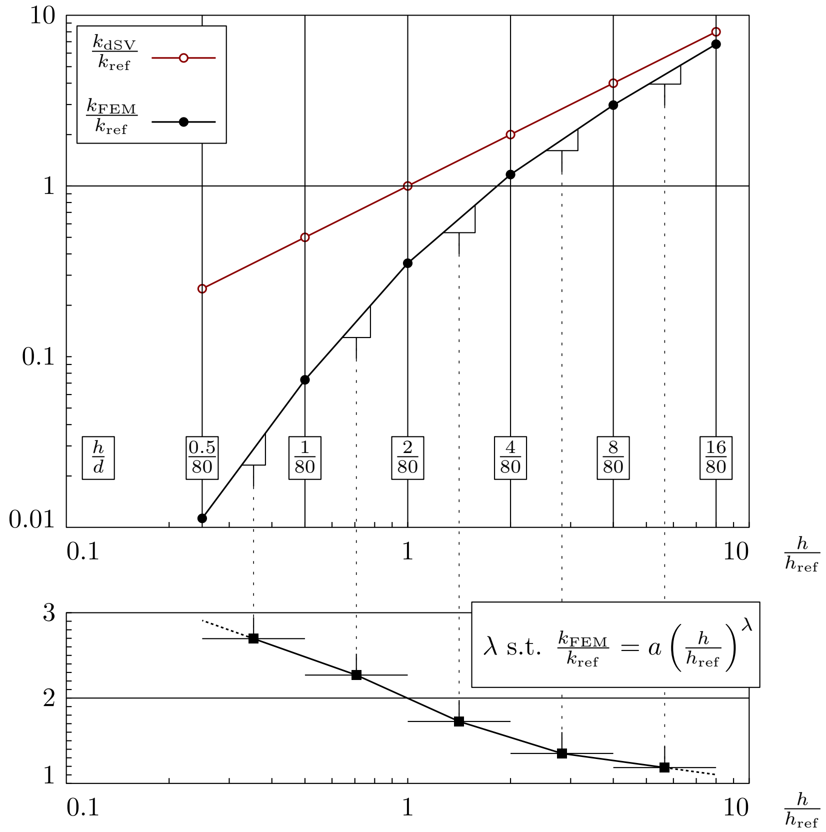


Figure 5: FIXME

$$f(\xi,\eta) \overset{\text{def}}{=} \sum_i N_i(\xi,\eta) f_i \tag{82}$$

$$N_i(\xi,\eta) \overset{\text{def}}{=} \frac{1}{4}\left(1 \pm \xi\right)\left(1 \pm \eta\right), \tag{83}$$

$$\frac{\partial f}{\partial \xi} = \underbrace{\left(\frac{f_2 - f_1}{2}\right)}_{[\Delta f/\Delta \xi]_{12}} \underbrace{\left(\frac{1-\eta}{2}\right)}_{N_1+N_2} + \underbrace{\left(\frac{f_3 - f_4}{2}\right)}_{[\Delta f/\Delta \xi]_{43}} \underbrace{\left(\frac{1+\eta}{2}\right)}_{N_4+N_3} = a\eta + b \tag{84}$$

$$\frac{\partial f}{\partial \eta} = \left(\frac{f_4 - f_1}{2}\right)\left(\frac{1-\xi}{2}\right) + \left(\frac{f_3 - f_2}{2}\right)\left(\frac{1+\xi}{2}\right) = c\xi + d. \tag{85}$$

$$\begin{aligned} f(\xi,\eta) &= \begin{bmatrix} N_1(\xi,\eta) & \cdots & N_i(\xi,\eta) & \cdots & N_n(\xi,\eta) \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} \\ &= \underline{\underline{\mathrm{N}}}\,(\xi,\eta)\,\underline{\mathrm{f}}\,, \end{aligned} \tag{86}$$

$$\underline{\mathrm{x}}\,\left(\underline{\xi}\right) = \underline{\mathrm{m}}\,\left(\underline{\xi}\right) = \sum_{i=1}^{4} N_i\left(\underline{\xi}\right)\,\underline{\mathrm{x}}_{\,i}, \tag{87}$$

$$\underline{\mathrm{m}}\,\left(\underline{\xi}\right) = \begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix}$$

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta) x_i \qquad y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta) y_i.$$

$$f(\xi,\eta) \overset{\text{def}}{=} \sum_i N_i(\xi,\eta) f_i \tag{88}$$

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underline{\underline{\mathrm{J}}}^{\top}(\xi,\eta;\underline{\mathrm{x}}_{\,i})} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \tag{89}$$

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \sum_i \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} f_i. \tag{90}$$

$$\underline{\underline{\mathrm{J}}}^{\top}(\xi, \eta) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{91}$$

$$= \sum_i \left(\begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 \\ \frac{\partial N_i}{\partial \eta} & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 & \frac{\partial N_i}{\partial \xi} \\ 0 & \frac{\partial N_i}{\partial \eta} \end{bmatrix} y_i \right) \tag{92}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \left(\underline{\underline{\mathrm{J}}}^{\top} \right)^{-1} \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial \xi} & \cdots \\ \cdots & \frac{\partial N_i}{\partial \eta} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ f_i \\ \vdots \end{bmatrix} \tag{93}$$

$$= \underbrace{\left(\underline{\underline{\mathrm{J}}}^{\top} \right)^{-1} \begin{bmatrix} \frac{\partial \underline{\mathrm{N}}}{\partial \xi} \\ \frac{\partial \underline{\mathrm{N}}}{\partial \eta} \end{bmatrix}}_{\underline{\underline{\mathrm{L}}}(\xi,\eta;\underline{\mathrm{x}}_i),\text{ or just }\underline{\underline{\mathrm{L}}}(\xi,\eta)} \quad \underline{\mathrm{f}} \tag{94}$$

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i; \tag{95}$$

$$p(\xi) \stackrel{\text{def}}{=} a_m \xi^m + a_{m-1} \xi^{m-1} + \ldots + a_1 \xi + a_0$$

$$\int_{-1}^{1} p(\xi) d\xi = \sum_{j=0}^{m} \frac{(-1)^j + 1}{j+1} a_j$$

$$r\left(a_j, (\xi_i, w_i)\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} p(\xi_i) w_i - \int_{-1}^{1} p(\xi) d\xi \tag{96}$$

$$\left\{ \frac{\partial r\left(a_j, (\xi_i, w_i)\right)}{\partial a_j} = 0, \quad j = 0 \ldots m \right. \tag{97}$$

$$\int_a^b g(x) dx = \int_{-1}^{1} g\left(m(\xi)\right) \frac{dm}{d\xi} d\xi \approx \sum_{i=1}^{n} g\left(m(\xi_i)\right) \left. \frac{dm}{d\xi} \right|_{\xi=\xi_i} w_i. \tag{98}$$

$$m(x) = \underbrace{\left(\frac{1-\xi}{2} \right)}_{N_1} a + \underbrace{\left(\frac{1+\xi}{2} \right)}_{N_2} b.$$

$$\frac{dm}{d\xi} = \frac{dN_1}{d\xi}a + \frac{dN_2}{d\xi}b = \frac{b-a}{2}$$

$$\int_a^b g(x)dx \approx \frac{b-a}{2} \sum_{i=1}^{n} g\left(\frac{b+a}{2} + \frac{b-a}{2}\xi_i\right) w_i. \tag{99}$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \sum_{i=1}^{p}\sum_{j=1}^{q} f(\xi_i,\eta_j)\, w_i w_j \tag{100}$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \sum_{l=1}^{pq} f\left(\underline{\xi}_{\,l}\right) w_l \tag{101}$$

$$\underline{\xi}_{\,l} = (\xi_i,\eta_j)\,, \quad w_l = w_i w_j, \quad l = 1\ldots pq \tag{102}$$

$$\begin{aligned} dA_{xy} = \quad & \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P, \eta_P) & y(\xi_P, \eta_P) \\ 1 & x(\xi_P + d\xi, \eta_P) & y(\xi_P + d\xi, \eta_P) \\ 1 & x(\xi_P + d\xi, \eta_P + d\eta) & y(\xi_P + d\xi, \eta_P + d\eta) \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P + d\xi, \eta_P + d\eta) & y(\xi_P + d\xi, \eta_P + d\eta) \\ 1 & x(\xi_P, \eta_P + d\eta) & y(\xi_P, \eta_P + d\eta) \\ 1 & x(\xi_P, \eta_P) & y(\xi_P, \eta_P) \end{vmatrix}. \end{aligned} \tag{103}$$

$$\mathcal{A} = \frac{1}{2!}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad \mathcal{H} = \frac{1}{n!}\begin{vmatrix} 1 & \underline{x}_{\,1} \\ 1 & \underline{x}_{\,2} \\ \vdots & \vdots \\ 1 & \underline{x}_{\,n+1} \end{vmatrix} \tag{104}$$

$$\begin{aligned} dA_{xy} \approx \quad & \frac{1}{2!}\begin{vmatrix} 1 & x & y \\ 1 & x + x_{,\xi}d\xi & y + y_{,\xi}d\xi \\ 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \\ 1 & x + x_{,\eta}d\eta & y + y_{,\eta}d\eta \\ 1 & x & y \end{vmatrix} \end{aligned}$$

$$dA_{xy} = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2}\begin{vmatrix} 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \\ 1 & x & y \end{vmatrix} d\xi d\eta$$

$$dA_{xy} = \frac{1}{2}\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2}\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta$$

$$dA_{xy} = \underbrace{\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix}}_{\left|J^{\mathrm{T}}(\xi_P,\eta_P;\underline{\mathrm{x}},\underline{\mathrm{y}})\right|} dA_{\xi\eta} \tag{105}$$

$$\iint_{A_{xy}} g(x,y)dA_{xy} = \int_{-1}^{1}\int_{-1}^{1} g\left(x\left(\xi,\eta\right),y\left(\xi,\eta\right)\right)\left|J(\xi,\eta)\right| d\xi d\eta, \tag{106}$$

$$\iint_{A_{xy}} g(\underline{\mathrm{x}})dA_{xy} \approx \sum_{l=1}^{pq} g\left(\underline{\mathrm{x}}\left(\underline{\xi}_l\right)\right)\left|J(\underline{\xi}_l)\right| w_l \tag{107}$$

$$dA_{xyz} = \sqrt{\begin{vmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{vmatrix}^2 + \begin{vmatrix} y_{,\xi} & y_{,\eta} \\ z_{,\xi} & z_{,\eta} \end{vmatrix}^2 + \begin{vmatrix} z_{,\xi} & z_{,\eta} \\ x_{,\xi} & x_{,\eta} \end{vmatrix}^2} d\xi d\eta \tag{108}$$

$$\underline{\mathrm{L}}\left(\xi,\eta;\underline{\mathrm{x}}_i\right) \approx ... \tag{109}$$

This is a four-node, thick-shell element with global displacements and rotations as degrees of freedom. Bilinear interpolation is used for the coordinates, displacements and the rotations. The membrane strains are obtained from the displacement field; the curvatures from the rotation field. The transverse shear strains are calculated at the middle of the edges and interpolated to the integration points. In this way, a very efficient and simple element is obtained which exhibits correct behavior in the limiting case of thin shells. The element can be used in curved shell analysis as well as in the analysis of complicated plate structures. For the latter case, the element is easy to use since connections between intersecting plates can be modeled without tying. Due to its simple formulation when compared to the standard higher order shell elements, it is less expensive and, therefore, very attractive in nonlinear analysis. The element is not very sensitive to distortion, particularly if the corner nodes lie in the same plane. All constitutive relations can be used with this element.

— MSC.Marc 2013.1 Documentation, vol. B, Element library.

$$\begin{bmatrix} X(\xi,\eta) \\ Y(\xi,\eta) \\ Z(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{n} N_i(\xi,\eta) \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}, \quad \begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \\ z(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{n} N_i(\xi,\eta) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{110}$$

$$\begin{bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \\ w(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{111}$$

$$\begin{bmatrix} \theta(\xi,\eta) \\ \phi(\xi,\eta) \\ \psi(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} \theta_i \\ \phi_i \\ \psi_i \end{bmatrix} \tag{112}$$

$$\underline{\mathrm{u}} = \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix} \qquad \underline{\mathrm{v}} = \begin{bmatrix} \vdots \\ v_i \\ \vdots \end{bmatrix} \qquad \underline{\mathrm{w}} = \begin{bmatrix} \vdots \\ w_i \\ \vdots \end{bmatrix}$$

$$\underline{\theta} = \begin{bmatrix} \vdots \\ \theta_i \\ \vdots \end{bmatrix} \qquad \underline{\phi} = \begin{bmatrix} \vdots \\ \phi_i \\ \vdots \end{bmatrix} \qquad \underline{\psi} = \begin{bmatrix} \vdots \\ \psi_i \\ \vdots \end{bmatrix}$$

$$u(\xi,\eta) = \underline{\underline{\mathrm{N}}}\,(\xi,\eta)\,\underline{\mathrm{u}} \qquad v(\xi,\eta) = \underline{\underline{\mathrm{N}}}\,(\xi,\eta)\,\underline{\mathrm{v}}$$

$$\underline{\mathrm{d}}^{\top} = \begin{bmatrix} \underline{\mathrm{u}}^{\top} & \underline{\mathrm{v}}^{\top} & \underline{\mathrm{w}}^{\top} & \underline{\theta}^{\top} & \underline{\phi}^{\top} & \underline{\psi}^{\top} \end{bmatrix} \tag{113}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \underbrace{\left(\underline{\underline{\mathrm{J}}}\,'\right)^{-1} \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial \xi} & \cdots \\ \cdots & \frac{\partial N_i}{\partial \eta} & \cdots \end{bmatrix}}_{\underline{\underline{\mathrm{L}}}\,(\xi,\eta;\,\underline{\mathrm{x}}_{\,i}) \text{ or just } \underline{\underline{\mathrm{L}}}\,(\xi,\eta)} \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix} \tag{114}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{L}}}\,(\xi,\eta) & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\mathrm{L}}}\,(\xi,\eta) \end{bmatrix}}_{\underline{\underline{\mathrm{Q}}}\,(\xi,\eta)} \begin{bmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{v}} \end{bmatrix} \tag{115}$$

$$\begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underline{\underline{\mathrm{Q}}}\,(\xi,\eta) \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{116}$$

$$\begin{bmatrix} e_x \\ e_y \\ g_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & +1 & 0 \end{bmatrix}}_{\underline{\underline{\mathrm{H}}}\,'} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underline{\underline{\mathrm{H}}}\,'\,\underline{\underline{\mathrm{Q}}}\,(\xi,\eta) \begin{bmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{v}} \end{bmatrix} \tag{117}$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & +1 \end{bmatrix}}_{\underline{\underline{\mathrm{H}}}\,''} \begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underline{\underline{\mathrm{H}}}\,''\,\underline{\underline{\mathrm{Q}}}\,(\xi,\eta) \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{118}$$

$$\underline{\mathrm{e}} = \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{H}}}'\,\underline{\underline{\mathrm{Q}}}(\xi,\eta) & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{\mathrm{B}}}_e(\xi,\eta)} \underline{\mathrm{d}} \tag{119}$$

$$\underline{\kappa} = \underbrace{\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\mathrm{H}}}''\,\underline{\underline{\mathrm{Q}}}(\xi,\eta) & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)} \underline{\mathrm{d}}\,. \tag{120}$$

$$\underline{\epsilon}\,(\xi,\eta,z) = \left(\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + z\,\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\right)\underline{\mathrm{d}}\,; \tag{121}$$

$$\begin{bmatrix} g_{zx} \\ g_{yz} \end{bmatrix} = \underline{\underline{\mathrm{L}}}(\xi,\eta)\,\underline{\mathrm{w}} + \begin{bmatrix} \underline{\underline{0}} & +\underline{\underline{\mathrm{N}}}(\xi,\eta) \\ -\underline{\underline{\mathrm{N}}}(\xi,\eta) & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix}, \tag{122}$$

$$\begin{bmatrix} g_{zx} \\ g_{yz} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\mathrm{L}}}(\xi,\eta) & \begin{matrix} 0 \\ -\underline{\underline{\mathrm{N}}}(\xi,\eta) \end{matrix} & \begin{matrix} \underline{\underline{\mathrm{N}}}(\xi,\eta) \\ 0 \end{matrix} & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{\mathrm{B}}}_\gamma(\xi,\eta)} \underline{\mathrm{d}} \tag{123}$$

$$\underline{\mathrm{d}}^\top = \begin{bmatrix} \underline{\mathrm{u}}^\top & \underline{\mathrm{v}}^\top & \underline{\mathrm{w}}^\top & \underline{\theta}^\top & \underline{\phi}^\top & \underline{\psi}^\top \end{bmatrix} \tag{124}$$

$$\underline{\mathrm{G}}^\top = \begin{bmatrix} \underline{\mathrm{U}}^\top & \underline{\mathrm{V}}^\top & \underline{\mathrm{W}}^\top & \underline{\Theta}^\top & \underline{\Phi}^\top & \underline{\Psi}^\top \end{bmatrix} \tag{125}$$

$$\delta\,\Upsilon_{\mathrm{e}} = \delta\,\underline{\mathrm{d}}^\top\,\underline{\mathrm{G}}\,. \tag{126}$$

$$\underline{\sigma} = \underline{\underline{\mathrm{D}}}(z)\left(\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)z\right)\underline{\mathrm{d}} \tag{127}$$

$$\delta\,\underline{\epsilon} = \left(\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)z\right)\delta\,\underline{\mathrm{d}} \tag{128}$$

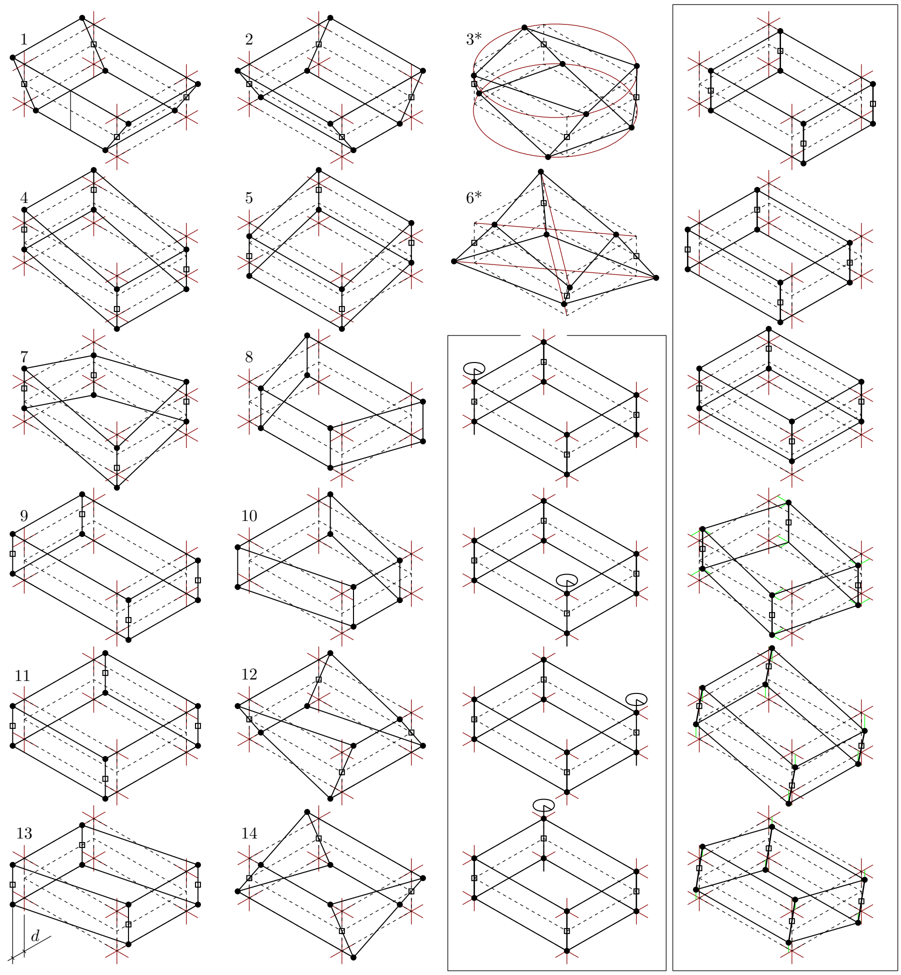
$$\underline{\mathrm{q}} = \left(\underline{\underline{\mathrm{a}}}\,\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{b}}}\,\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\right)\underline{\mathrm{d}} \tag{129}$$

$$\underline{\mathrm{m}} = \left(\underline{\underline{\mathrm{b}}}^\top\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{c}}}\,\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\right)\underline{\mathrm{d}}\,, \tag{130}$$

$$\delta\,\underline{\mathrm{e}} = \underline{\underline{\mathrm{B}}}_e(\xi,\eta)\,\delta\,\underline{\mathrm{d}} \tag{131}$$

$$\delta\,\underline{\kappa} = \underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\,\delta\,\underline{\mathrm{d}}\,, \tag{132}$$

$$\begin{aligned}
\delta\,\Upsilon_{\mathrm{i}}^\dagger &= \iint_{\mathcal{A}}\int_h \delta\,\underline{\epsilon}^\top\,\underline{\sigma}\,dzd\mathcal{A} \\
&= \iint_{\mathcal{A}}\int_h \left(\left(\underline{\underline{\mathrm{B}}}_e + \underline{\underline{\mathrm{B}}}_\kappa z\right)\delta\,\underline{\mathrm{d}}\right)^\top \underline{\underline{\mathrm{D}}}\left(\underline{\underline{\mathrm{B}}}_e + \underline{\underline{\mathrm{B}}}_\kappa z\right)\underline{\mathrm{d}}\,dzd\mathcal{A} \\
&= \delta\,\underline{\mathrm{d}}^\top\left[\iint_{\mathcal{A}}\int_h \left(\underline{\underline{\mathrm{B}}}_e^\top + \underline{\underline{\mathrm{B}}}_\kappa^\top z\right)\underline{\underline{\mathrm{D}}}\left(\underline{\underline{\mathrm{B}}}_e + \underline{\underline{\mathrm{B}}}_\kappa z\right)dzd\mathcal{A}\right]\underline{\mathrm{d}} \\
&= \delta\,\underline{\mathrm{d}}^\top\,\underline{\underline{\mathrm{K}}}^\dagger\,\underline{\mathrm{d}}\,,
\end{aligned} \tag{133}$$



rectangular plate element $2a \cdot 2b$, thickness h
nodal displacements magnitude d,
$x = a\xi$, $y = b\eta$, $-\frac{h}{2} \leq z \leq \frac{h}{2}$

$$
\begin{aligned}
\delta\,\Upsilon_{\rm i}^{\dagger} &= \iint_{\mathcal{A}} \left(\delta\,\underline{\rm e}^{\top}\,\underline{\rm q} + \delta\,\underline{\kappa}^{\top}\,\underline{\rm m}\right) d\mathcal{A} \\
&= \delta\,\underline{\rm d}^{\top} \left[\iint_{\mathcal{A}} \begin{bmatrix} \underline{\underline{\rm B}}_{e} \\ \underline{\underline{\rm B}}_{\kappa} \end{bmatrix}^{\top} \begin{bmatrix} \underline{\underline{\rm a}} & \underline{\underline{\rm b}} \\ \underline{\underline{\rm b}}^{\rm T} & \underline{\underline{\rm c}} \end{bmatrix} \begin{bmatrix} \underline{\underline{\rm B}}_{e} \\ \underline{\underline{\rm B}}_{\kappa} \end{bmatrix} d\mathcal{A}\right] \underline{\rm d} \\
&= \delta\,\underline{\rm d}^{\top}\,\underline{\underline{\rm K}}^{\dagger}\,\underline{\rm d}\,,
\end{aligned} \tag{134}
$$

$$
\left\{\underline{\underline{\rm a}}\,,\,\underline{\underline{\rm b}}\,,\,\underline{\underline{\rm c}}\right\} = \int_{h} \underline{\underline{\rm D}}\,\left\{1, z, z^{2}\right\} dz,
$$

$$
\iiint_{\Omega} g(\xi,\eta,x,y,z)d\Omega = \tag{135}
$$

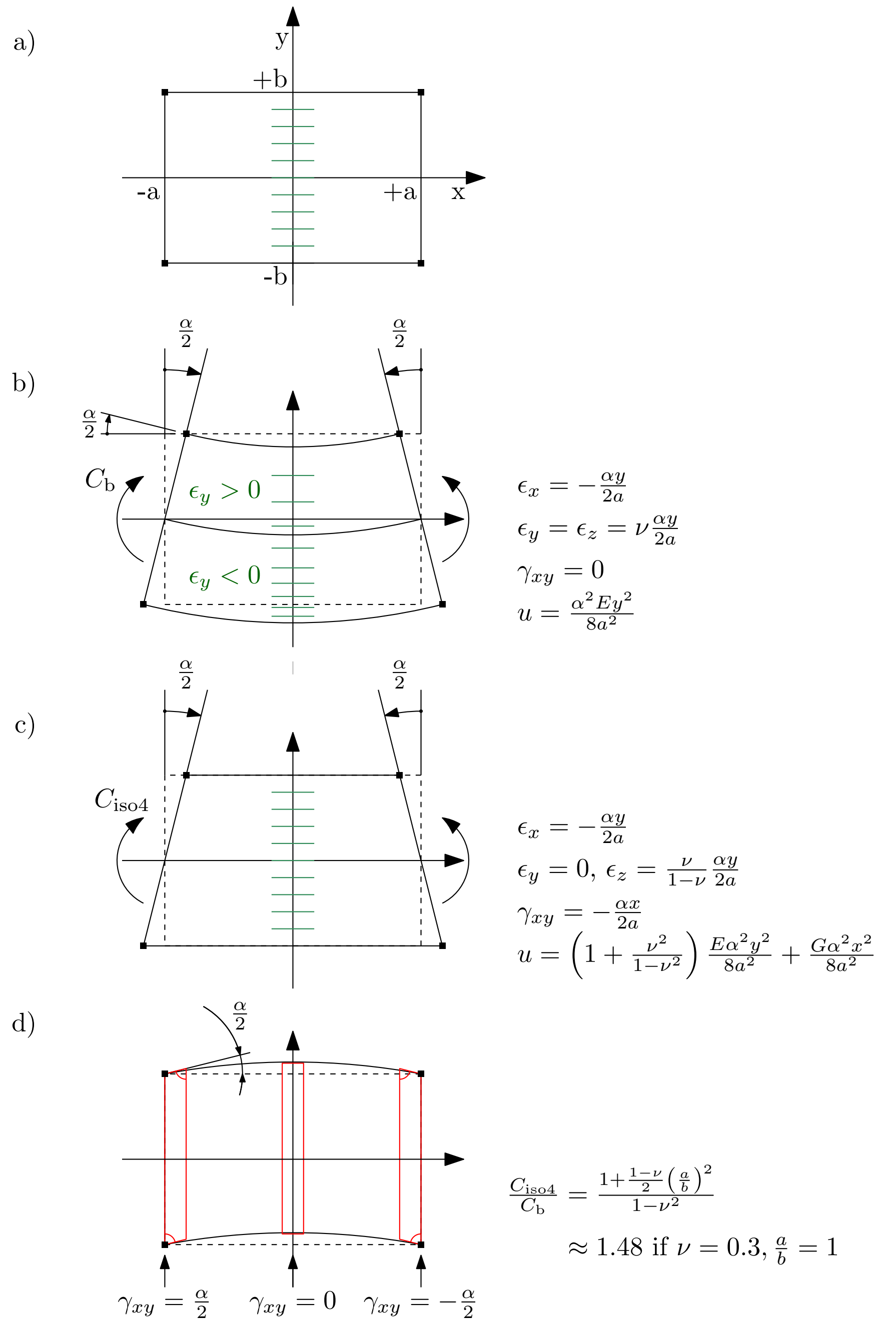
$$
= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-\frac{h}{2}+o}^{+\frac{h}{2}+o} g(\xi,\eta,x(\xi,\eta),y(\xi,\eta),z)dz\left|\underline{\underline{\rm J}}\,(\xi,\eta)\right| d\xi d\eta,
$$

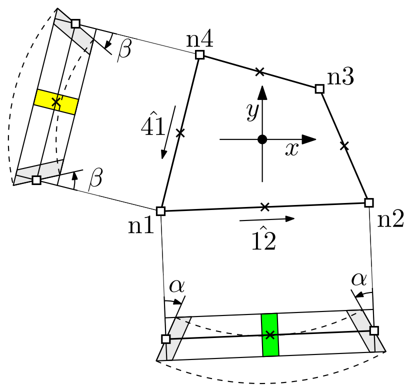
$$
\begin{aligned}
\delta\,\Upsilon_{\rm i}^{\ddagger} &= \iint_{\mathcal{A}} \delta\,\underline{\gamma}_{z}^{\top}\,\underline{\rm q}_{z} d\mathcal{A} \\
&= \delta\,\underline{\rm d}^{\top}\left[\iint_{\mathcal{A}} \underline{\underline{\rm B}}_{\gamma}^{\top}\,\underline{\underline{\Gamma}}\,\underline{\underline{\rm B}}_{\gamma} d\mathcal{A}\right] \underline{\rm d} \\
&= \delta\,\underline{\rm d}^{\top}\,\underline{\underline{\rm K}}^{\ddagger}\,\underline{\rm d}\,.
\end{aligned} \tag{136}
$$

$$
\begin{aligned}
\delta\,\Upsilon_{\rm i} &= \delta\,\Upsilon_{\rm i}^{\dagger} + \delta\,\Upsilon_{\rm i}^{\ddagger} \\
&= \delta\,\underline{\rm d}^{\top}\left(\underline{\underline{\rm K}}^{\dagger} + \underline{\underline{\rm K}}^{\ddagger}\right)\underline{\rm d} \\
&= \delta\,\underline{\rm d}^{\top}\,\underline{\underline{\rm K}}\,\underline{\rm d}\,.
\end{aligned} \tag{137}
$$

$$
\delta\,\underline{\rm d}^{\top}\,\underline{\rm G} = \delta\,\Upsilon_{\rm e} = \delta\,\Upsilon_{\rm i} = \delta\,\underline{\rm d}^{\top}\,\underline{\underline{\rm K}}\,\underline{\rm d}\,, \quad \forall\,\delta\,\underline{\rm d}\,, \tag{138}
$$

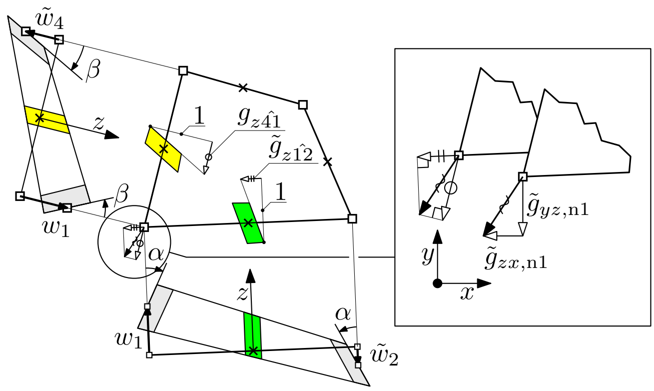
$$
\underline{\rm G} = \underline{\underline{\rm K}}\,\underline{\rm d}\,; \tag{139}
$$


a)
y
+b
-a
+a
x
-b
b)
$\frac{\alpha}{2}$
C_b
$\epsilon_y > 0$
$\epsilon_y < 0$
$\epsilon_x = -\frac{\alpha y}{2a}$
$\epsilon_y = \epsilon_z = \nu \frac{\alpha y}{2a}$
$\gamma_{xy} = 0$
$u = \frac{\alpha^2 E y^2}{8a^2}$
c)
C_iso4
$\epsilon_x = -\frac{\alpha y}{2a}$
$\epsilon_y = 0,\ \epsilon_z = \frac{\nu}{1-\nu} \frac{\alpha y}{2a}$
$\gamma_{xy} = -\frac{\alpha x}{2a}$
$u = \left(1 + \frac{\nu^2}{1-\nu^2}\right) \frac{E\alpha^2 y^2}{8a^2} + \frac{G\alpha^2 x^2}{8a^2}$
d)
$\frac{C_\mathrm{iso4}}{C_\mathrm{b}} = \frac{1+\frac{1-\nu}{2}\left(\frac{a}{b}\right)^2}{1-\nu^2}$
≈ 1.48 if $\nu = 0.3,\ \frac{a}{b} = 1$
$\gamma_{xy} = \frac{\alpha}{2}$
$\gamma_{xy} = 0$
$\gamma_{xy} = -\frac{\alpha}{2}$



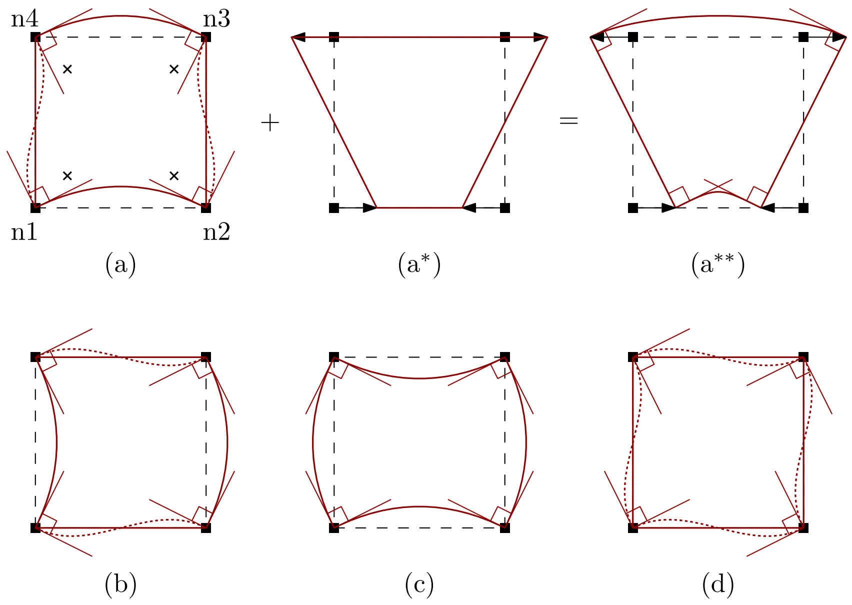
β
n4
n3
$\hat{41}$
y
x
β
n1
n2
$\hat{12}$
α
α

(a)



$\tilde{w}_4$
β
z
1
$g_{z\hat{4}\hat{1}}$
$\tilde{g}_{z\hat{1}\hat{2}}$
1
β
w_1
α
z
α
w_1
$\tilde{w}_2$
$\tilde{g}_{yz,\text{n1}}$
y
$\tilde{g}_{zx,\text{n1}}$
x

(b)


n4
n3
n1
n2
+
=
(a)
(a*)
(a**)
(b)
(c)
(d)

$$\underline{\underline{\mathrm{S}}}(\xi,\eta,z) = \begin{bmatrix} \dots & \hat{u}_i(\xi,\eta,z) & \dots \\ \dots & \hat{v}_i(\xi,\eta,z) & \dots \\ \dots & \hat{w}_i(\xi,\eta,z) & \dots \end{bmatrix} \tag{140}$$

$$\underline{\mathrm{u}}(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}(\xi,\eta,z)\,\underline{\mathrm{d}}\,. \tag{141}$$

$$\underline{\dot{\mathrm{u}}}(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}(\xi,\eta,z)\,\underline{\dot{\mathrm{d}}} \tag{142}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \underline{\dot{\mathrm{u}}}^\top\,\underline{\dot{\mathrm{u}}}\,\rho d\Omega \tag{143}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \left[\underline{\underline{\mathrm{S}}}\,\underline{\dot{\mathrm{d}}}\right]^\top \left[\underline{\underline{\mathrm{S}}}\,\underline{\dot{\mathrm{d}}}\right]\rho d\Omega, \tag{144}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\,\underline{\dot{\mathrm{d}}}^\top \left[\iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\underline{\mathrm{S}}}\,\rho d\Omega\right]\underline{\dot{\mathrm{d}}} = \frac{1}{2}\,\underline{\dot{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\underline{\dot{\mathrm{d}}}\,. \tag{145}$$

$$\underline{\underline{\mathrm{M}}} = \iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\underline{\mathrm{S}}}\,\rho d\Omega, \tag{146}$$

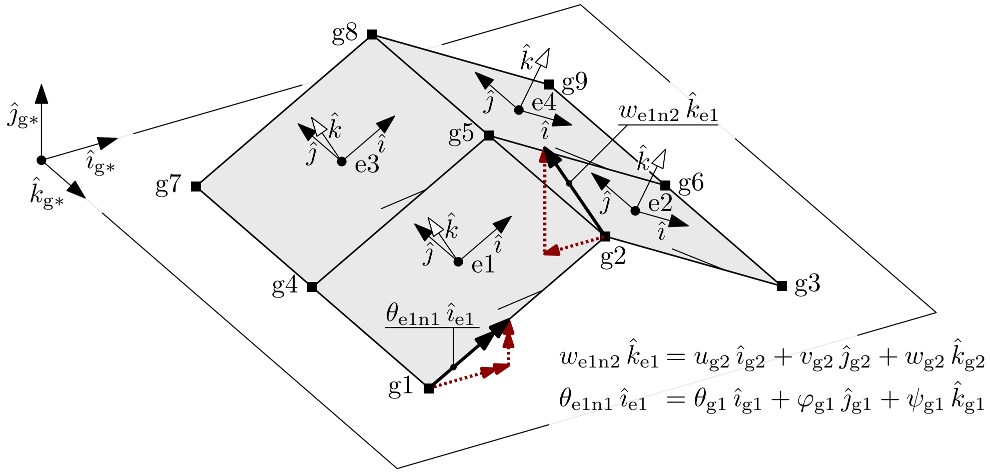
$$\begin{aligned}
\underline{\dot{\mathrm{d}}}^\top\,\underline{\mathrm{G}} = \frac{dE_{\mathrm{kin}}}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\,\underline{\dot{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\underline{\dot{\mathrm{d}}}\right) \\
&= \frac{1}{2}\left(\underline{\ddot{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\underline{\dot{\mathrm{d}}} + \underline{\dot{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\underline{\ddot{\mathrm{d}}}\right) \\
&= \underline{\dot{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\underline{\ddot{\mathrm{d}}}\,.
\end{aligned}$$

$$\underline{\mathrm{G}} = \underline{\underline{\mathrm{M}}}\,\underline{\ddot{\mathrm{d}}} \tag{147}$$

$$\delta\,\underline{\mathrm{u}}(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}(\xi,\eta,z)\delta\,\underline{\mathrm{d}}\,, \tag{148}$$

$$\begin{aligned}
\delta\,\underline{\mathrm{d}}^\top\,\underline{\mathrm{F}} &= \iiint_\Omega (\delta\,\underline{\mathrm{u}})^\top\,\underline{\mathrm{p}}\,d\Omega \\
&= \iiint_\Omega \left(\underline{\underline{\mathrm{S}}}\,\delta\,\underline{\mathrm{d}}\right)^\top\,\underline{\mathrm{p}}\,d\Omega \\
&= \delta\,\underline{\mathrm{d}}^\top \iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\mathrm{p}}\,d\Omega,
\end{aligned}$$

$$\underline{\mathrm{F}} = \iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\mathrm{p}}\,d\Omega \tag{149}$$



$$\underline{\mathrm{G}}_{\,\mathrm{e}j} = \underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{e}j} \tag{150}$$

$$\underline{\mathrm{d}}_{\,\mathrm{g}l} = \begin{bmatrix} u_{\mathrm{g}l} \\ v_{\mathrm{g}l} \\ w_{\mathrm{g}l} \\ \theta_{\mathrm{g}l} \\ \varphi_{\mathrm{g}l} \\ \psi_{\mathrm{g}l} \end{bmatrix}. \tag{151}$$

$$\underline{\mathrm{d}}_{\,\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{d}}_{\,\mathrm{g1}}^{\top} & \underline{\mathrm{d}}_{\,\mathrm{g2}}^{\top} & \cdots & \underline{\mathrm{d}}_{\,\mathrm{g}l}^{\top} & \cdots & \underline{\mathrm{d}}_{\,\mathrm{g}n}^{\top} \end{bmatrix} \tag{152}$$

$$\underline{\mathrm{F}}_{\,\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{F}}_{\,\mathrm{g1}}^{\top} & \underline{\mathrm{F}}_{\,\mathrm{g2}}^{\top} & \cdots & \underline{\mathrm{F}}_{\,\mathrm{g}l}^{\top} & \cdots & \underline{\mathrm{F}}_{\,\mathrm{g}n}^{\top} \end{bmatrix}; \tag{153}$$

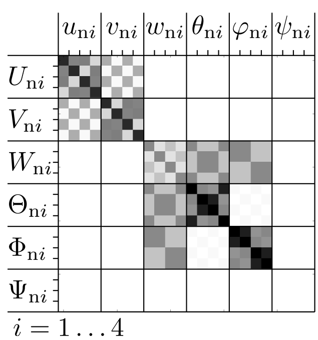
$$\underline{\mathrm{R}}_{\,\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{R}}_{\,\mathrm{g1}}^{\top} & \underline{\mathrm{R}}_{\,\mathrm{g2}}^{\top} & \cdots & \underline{\mathrm{R}}_{\,\mathrm{g}l}^{\top} & \cdots & \underline{\mathrm{R}}_{\,\mathrm{g}n}^{\top} \end{bmatrix} \tag{154}$$

$$w_{\mathrm{e1n2}} = \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle u_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle v_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle w_{\mathrm{g2}} \tag{155}$$

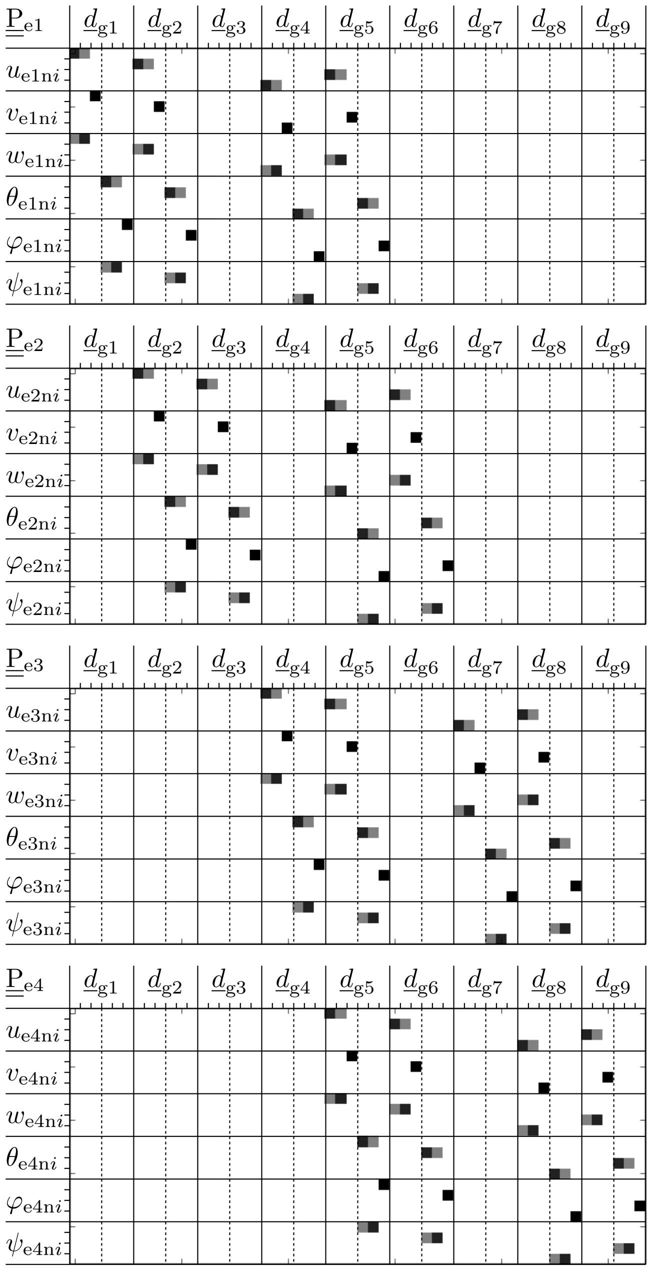
$$\theta_{\mathrm{e1n1}} = \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \theta_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \phi_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle \psi_{\mathrm{g1}} \tag{156}$$

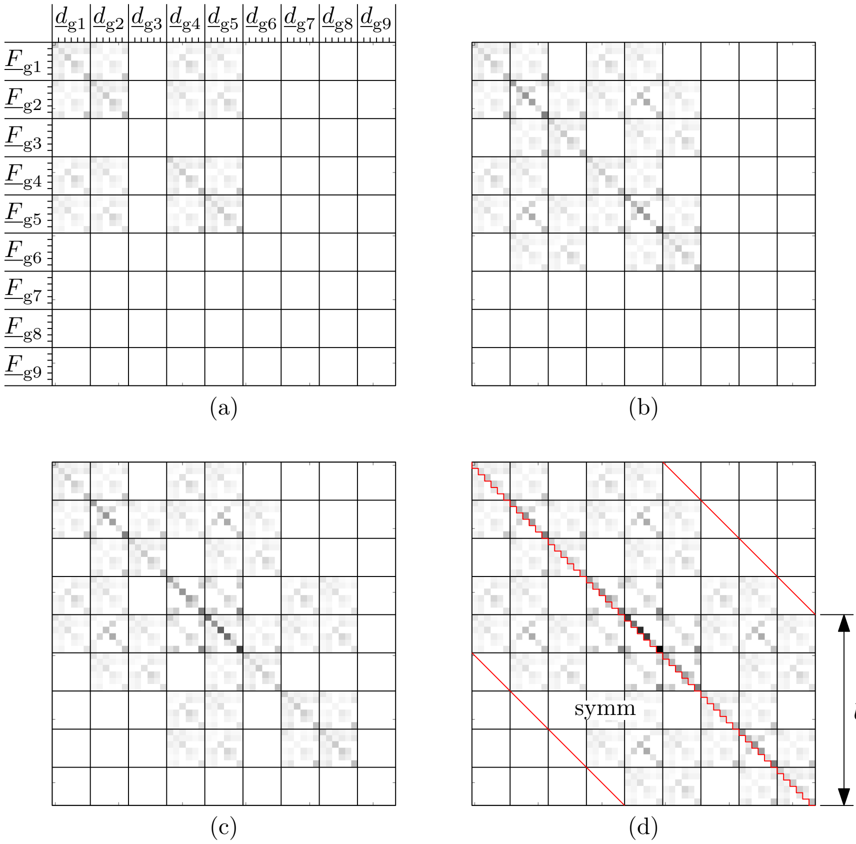
$$\begin{aligned}
\left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{10,7} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{13,4} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{10,8} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{13,5} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{10,9} &= \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{13,6} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle,
\end{aligned}$$

node	X	Y	Z
g1	$-ac$	0	$+a$
g2	0	$+as$	$+a$
g3	$+ac$	0	$+a$
g4	$-ac$	0	0
g5	0	$+as$	0
g6	$+ac$	0	0
g7	$-ac$	0	$-a$
g8	0	$+as$	$-a$
g9	$+ac$	0	$-a$



	n1	n2	n3	n4
e1	g1	g2	g5	g4
e2	g2	g3	g6	g5
e3	g4	g5	g8	g7
e4	g5	g6	g9	g8


Pe1
dg1 dg2 dg3 dg4 dg5 dg6 dg7 dg8 dg9
ue1ni
ve1ni
we1ni
θe1ni
φe1ni
ψe1ni
Pe2
dg1 dg2 dg3 dg4 dg5 dg6 dg7 dg8 dg9
ue2ni
ve2ni
we2ni
θe2ni
φe2ni
ψe2ni
Pe3
dg1 dg2 dg3 dg4 dg5 dg6 dg7 dg8 dg9
ue3ni
ve3ni
we3ni
θe3ni
φe3ni
ψe3ni
Pe4
dg1 dg2 dg3 dg4 dg5 dg6 dg7 dg8 dg9
ue4ni
ve4ni
we4ni
θe4ni
φe4ni
ψe4ni



$$\underline{\mathrm{d}}_{\,\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \quad \forall j. \tag{157}$$

$$\underline{\mathrm{G}}_{\,\mathrm{e}j} = \underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \quad \forall j; \tag{158}$$

$$\delta\,\underline{\mathrm{d}}_{\,\mathrm{g}}^{\top}\,\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \left(\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\delta\,\underline{\mathrm{d}}_{\,\mathrm{g}}\right)^{\top}\,\underline{\mathrm{G}}_{\,\mathrm{e}j}, \quad \forall\ \delta\,\underline{\mathrm{d}}_{\,\mathrm{g}} \tag{159}$$

$$\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\mathrm{G}}_{\,\mathrm{e}j} \tag{160}$$

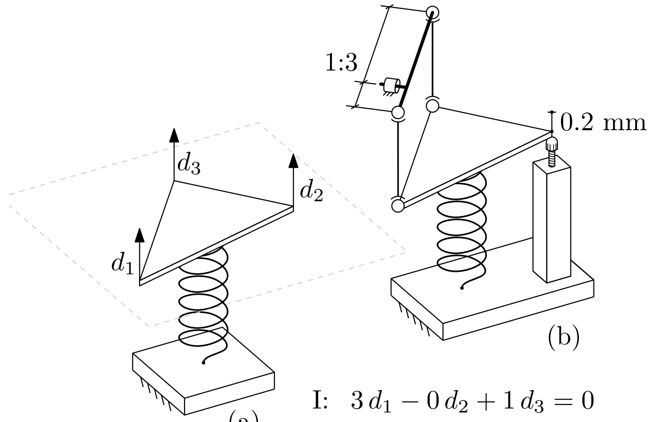
$$\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}; \tag{161}$$

$$\underline{\mathrm{G}}_{\,\mathrm{g}} = \sum_{j}\,\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \left(\sum_{j}\underbrace{\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}}_{\underline{\underline{\mathrm{K}}}_{\,\mathrm{g}\leftarrow\mathrm{e}j}}\right)\underline{\mathrm{d}}_{\,\mathrm{g}} = \underline{\underline{\mathrm{K}}}_{\,\mathrm{g}}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \tag{162}$$

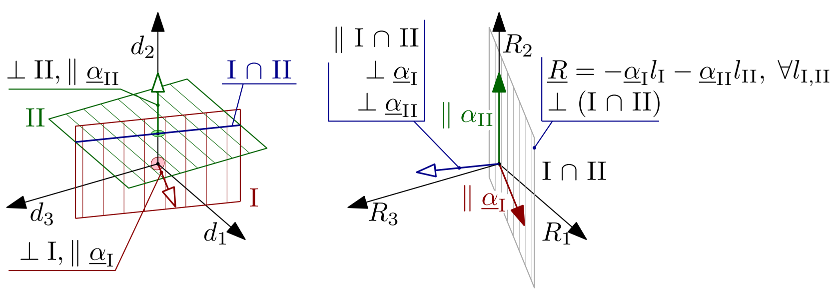
$$b_{\mathrm{e}j} = (i_{\max} - i_{\min} + 1)\, l, \tag{163}$$

$$b = \max_{\mathrm{e}j} b_{\mathrm{e}j} \tag{164}$$

$$\underline{\mathrm{F}}_{\,\mathrm{g}} = \sum_j \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top} \underline{\mathrm{F}}_{\,\mathrm{e}j}; \tag{165}$$



$$\text{I:}\quad 3\,d_1 - 0\,d_2 + 1\,d_3 = 0$$
$$\text{II:}\quad 0\,d_1 + 1\,d_2 + 0\,d_3 = 0.2$$



(a) configuration space (b) reaction space

$$\sum_i \alpha_{ji} d_i = \underline{\alpha}_j^\top \underline{\mathrm{d}} = \beta_j$$

$$\underline{\alpha}_{\mathrm{I}}^\top = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \qquad \beta_{\mathrm{I}} = 0$$
$$\underline{\alpha}_{\mathrm{II}}^\top = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad \beta_{\mathrm{II}} = 0.2$$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \qquad \underline{\Delta} = \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix}..$$

$$\sum_i \alpha_{ji} d_i = \underline{\alpha}_j^\top \underline{\mathrm{d}} = \beta_j, \quad j = 1 \dots m \tag{166}$$

$$\underline{\underline{\mathcal{L}}}^{\top} \underline{\mathrm{d}} = \underline{\beta} \, . \tag{167}$$

$$\underline{\underline{\mathcal{L}}}^{\top} \delta \underline{\mathrm{d}} = \underline{0} \, , \tag{168}$$

$$\underline{\mathrm{R}} = - \underline{\underline{\mathcal{L}}} \, \underline{\ell} \, , \tag{169}$$

$$\underline{\mathrm{R}}^{j} = - \begin{bmatrix} \vdots \\ \alpha_{ji} \\ \vdots \end{bmatrix} \ell_j \tag{170}$$

$$\underline{\underline{\mathrm{K}}} \, \underline{\mathrm{d}} = \underline{\mathrm{F}} + \underline{\mathrm{R}} \, . \tag{171}$$

$$\underline{\underline{\mathrm{K}}} \, \underline{\mathrm{d}} + \underline{\underline{\mathcal{L}}} \, \underline{\ell} = \underline{\mathrm{F}}$$

$$\begin{bmatrix} \underline{\underline{\mathrm{K}}} & \underline{\underline{\mathcal{L}}} \\ \underline{\underline{\mathcal{L}}}^{\top} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{d}} \\ \underline{\ell} \end{bmatrix} = \begin{bmatrix} \underline{\mathrm{F}} \\ \underline{\beta} \end{bmatrix} , \tag{172}$$

$$\frac{1}{2} \underline{\mathrm{d}}^{\top} \underline{\underline{\mathrm{K}}} \, \underline{\mathrm{d}} - \underline{\mathrm{d}}^{\top} \underline{\mathrm{F}} + \underline{\ell}^{\top} \left(\underline{\underline{\mathcal{L}}}^{\top} \underline{\mathrm{d}} - \underline{\beta} \right) , \tag{173}$$

$$\frac{1}{2} \underline{\mathrm{d}}^{\top} \underline{\underline{\mathrm{K}}} \, \underline{\mathrm{d}} - \underline{\mathrm{d}}^{\top} \underline{\mathrm{F}}$$

$$\underline{\underline{\mathcal{L}}}^{\top} \underline{\mathrm{d}} - \underline{\beta} = \underline{0}$$

$$\underline{\mathrm{R}} = - \underline{\underline{\mathcal{L}}} \, \underline{\ell}^{*} .$$

$$d_k \equiv t_j = \sum_{i \neq k} \left(- \frac{\alpha_{ji}}{\alpha_{jk}} \right) d_i + \left(\frac{\beta_j}{\alpha_{jk}} \right) , \quad j = 1 \ldots m \tag{174}$$

$$\underline{\underline{\mathrm{A}}} \, \underline{\mathrm{t}} + \left(- \underline{\underline{\mathrm{B}}} \right) \underline{\mathrm{r}} = \underline{\beta} \quad \Rightarrow \quad \underline{\underline{\mathrm{A}}} \, \underline{\mathrm{t}} = \underline{\underline{\mathrm{B}}} \, \underline{\mathrm{r}} + \underline{\beta} \, ,$$

$$\underline{\mathrm{t}} = \left[\underline{\underline{\mathrm{A}}}^{-1} \underline{\underline{\mathrm{B}}} \right] \underline{\mathrm{r}} + \left[\underline{\underline{\mathrm{A}}}^{-1} \underline{\beta} \right] , \tag{175}$$

$$t_j = \underline{\mathrm{a}}_j^{\top} \underline{\underline{\mathrm{B}}} \, \underline{\mathrm{r}} + \underline{\mathrm{a}}_j^{\top} \underline{\beta} \, , \quad j = 1 \ldots m \tag{176}$$

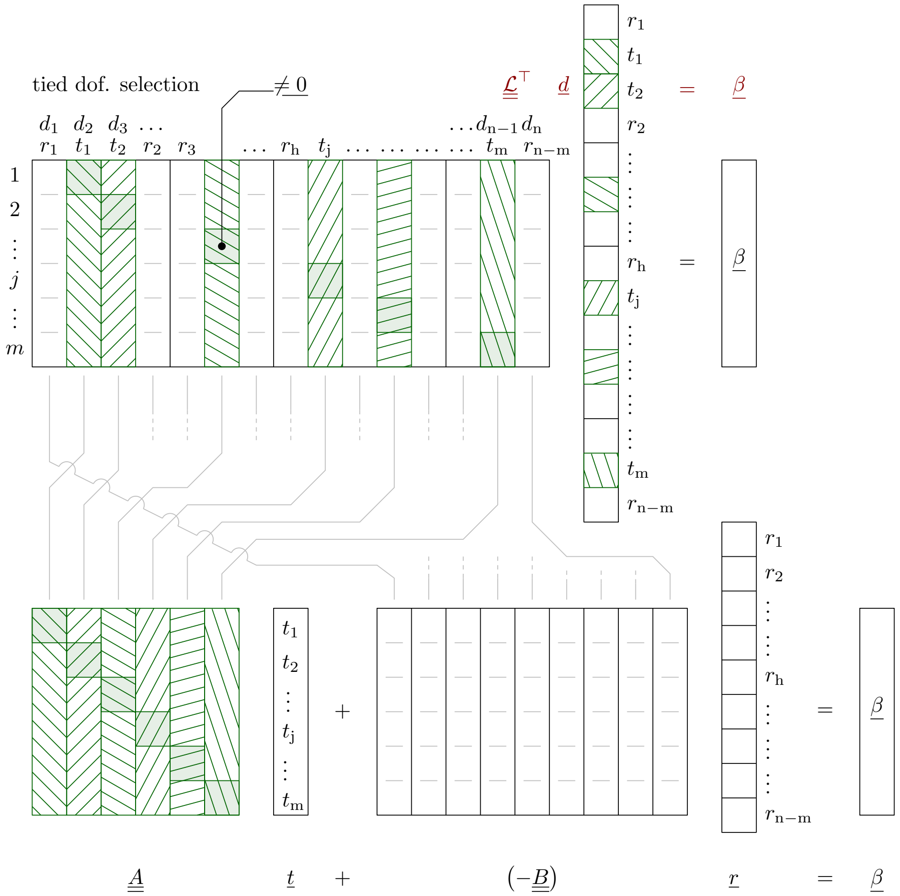
$$\underline{\mathrm{d}} = \underline{\underline{\Lambda}} \, \underline{\mathrm{r}} + \underline{\Delta} \, , \tag{177}$$

$$\delta \underline{\mathrm{d}} = \underline{\underline{\Lambda}} \, \delta \underline{\mathrm{r}} = \underline{\Lambda}_{1} \delta r_1 + \underline{\Lambda}_{2} \delta r_2 + \ldots + \underline{\Lambda}_{n-m} \delta r_{n-m} \tag{178}$$

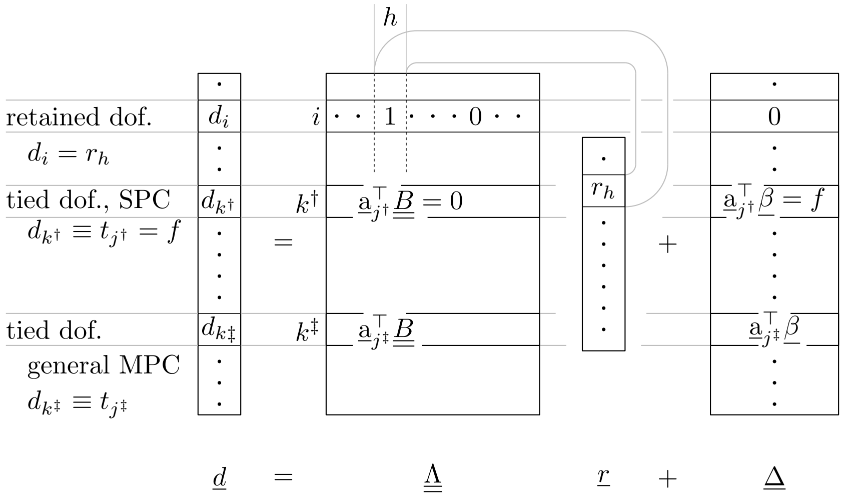
$$\left\langle \left[\underline{\underline{\Lambda}} \right]_{\mathrm{col}\ h} , \underline{\mathrm{R}} \right\rangle = 0 \quad h = 1 \ldots n - m, \tag{179}$$

$$\underline{\underline{\Lambda}}^{\top} \underline{\mathrm{R}} = \underline{0} \, . \tag{180}$$

$$\underline{\underline{\mathrm{K}}} \left(\underline{\underline{\Lambda}} \, \underline{\mathrm{r}} + \underline{\Delta} \right) = \underline{\mathrm{F}} + \underline{\mathrm{R}} \tag{181}$$



tied dof. selection
≠ 0
$\underline{\underline{\mathcal{L}}}^\top$ $\underline{d}$ = $\underline{\beta}$
d_1 d_2 d_3 … … d_{n-1} d_n
r_1 t_1 t_2 r_2 r_3 … r_h t_j … … … … t_m r_{n-m}
1 2 ⋮ j ⋮ m
r_1 t_1 t_2 r_2 ⋮ ⋮ ⋮ r_h t_j ⋮ ⋮ ⋮ ⋮ t_m r_{n-m}
= $\underline{\beta}$
t_1 t_2 ⋮ t_j ⋮ t_m
+
r_1 r_2 ⋮ ⋮ r_h ⋮ ⋮ ⋮ r_{n-m}
= $\underline{\beta}$
$\underline{\underline{A}}$ $\underline{t}$ + $(-\underline{\underline{B}})$ $\underline{r}$ = $\underline{\beta}$



$$\underline{\underline{\mathrm{K}}}\,\underline{\underline{\Lambda}}\,\underline{\mathrm{r}} = \left(\underline{\mathrm{F}} - \underline{\underline{\mathrm{K}}}\,\underline{\underline{\Delta}}\right) + \underline{\mathrm{R}}, \tag{182}$$

$$\underbrace{\underline{\underline{\Lambda}}^\top \underline{\underline{\mathrm{K}}}\,\underline{\underline{\Lambda}}}_{\underline{\underline{\mathrm{K}}}_{\mathrm{R}}}\,\underline{\mathrm{r}} = \underbrace{\underline{\underline{\Lambda}}^\top \left(\underline{\mathrm{F}} - \underline{\underline{\mathrm{K}}}\,\underline{\underline{\Delta}}\right)}_{\underline{\mathrm{F}}_{\mathrm{R}}} + \underbrace{\underline{\underline{\Lambda}}^\top \underline{\mathrm{R}}}_{=0}, \tag{183}$$

$$\underline{\underline{\mathrm{K}}}_{\mathrm{R}}\,\underline{\mathrm{r}} = \underline{\mathrm{F}}_{\mathrm{R}} \tag{184}$$

$$\underline{\mathrm{d}}^* = \underline{\underline{\Lambda}}\,\underline{\mathrm{r}}^* + \underline{\underline{\Delta}}; \tag{185}$$

$$\underline{\mathrm{R}}^* = \underline{\underline{\mathrm{K}}}\left(\underline{\underline{\Lambda}}\,\underline{\mathrm{r}}^* + \underline{\underline{\Delta}}\right) - \underline{\mathrm{F}}. \tag{186}$$

$$\underline{\mathrm{d}}^*_{\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\mathrm{e}j}\,\underline{\mathrm{d}}^*. \tag{187}$$

$$\underline{\mathrm{e}} = \underline{\underline{\mathrm{B}}}^{e}_{\mathrm{e}j}(\xi,\eta)\,\underline{\mathrm{d}}^*_{\mathrm{e}j} \qquad \underline{\kappa} = \underline{\underline{\mathrm{B}}}^{\kappa}_{\mathrm{e}j}(\xi,\eta)\,\underline{\mathrm{d}}^*_{\mathrm{e}j} \tag{188}$$

$$\underline{\epsilon} = \left(\underline{\underline{\mathrm{B}}}^{e}_{\mathrm{e}j}(\xi,\eta) + \underline{\underline{\mathrm{B}}}^{\kappa}_{\mathrm{e}j}(\xi,\eta)z\right)\,\underline{\mathrm{d}}^*_{\mathrm{e}j}. \tag{189}$$

$$\underline{\gamma}_z = \underline{\underline{\mathrm{B}}}^{\gamma}_{\mathrm{e}j}(\xi,\eta)\,\underline{\mathrm{d}}^*_{\mathrm{e}j}. \tag{190}$$

$$
\begin{bmatrix} u_i \\ v_i \\ w_i \\ \theta_i \\ \phi_i \\ \psi_i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & +(z_i - z_C) & -(y_i - y_C) \\ 0 & 1 & 0 & -(z_i - z_C) & 0 & +(x_i - x_C) \\ 0 & 0 & 1 & +(y_i - y_C) & -(x_i - x_C) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{\mathrm{L}}}_i} \cdot \begin{bmatrix} u_C \\ v_C \\ w_C \\ \theta_C \\ \phi_C \\ \psi_C \end{bmatrix} \tag{191}
$$

$$
\underline{\underline{\mathrm{J}}} = \sum_i q_i \begin{bmatrix} y_{Gi}^2 + z_{Gi}^2 & -x_{Gi}\, y_{Gi} & -x_{Gi}\, z_{Gi} \\ -y_{Gi}\, x_{Gi} & z_{Gi}^2 + x_{Gi}^2 & -y_{Gi}\, z_{Gi} \\ -z_{Gi}\, x_{Gi} & -z_{Gi}\, y_{Gi} & x_{Gi}^2 + y_{Gi}^2 \end{bmatrix}, \tag{192}
$$

$$
\begin{bmatrix} x_{Gi} \\ y_{Gi} \\ z_{Gi} \end{bmatrix} = \underline{\mathrm{x}}_{Gi} = \underline{\mathrm{x}}_i - \underline{\mathrm{x}}_G.
$$

$$
\begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} = q_i \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} \wedge \underline{\mathrm{x}}_{Gi} \right) \tag{193}
$$

$$
\underbrace{\begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix}}_{\underline{\mathrm{U}}_i} = q_i \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & +z_{Gi} & -y_{Gi} \\ 0 & 1 & 0 & -z_{Gi} & 0 & +x_{Gi} \\ 0 & 0 & 1 & +y_{Gi} & -x_{Gi} & 0 \end{bmatrix}}_{\underline{\underline{\mathrm{S}}}_i} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}}_{\underline{\mathrm{a}}}, \tag{194}
$$

$$
\begin{bmatrix} \check{U}_C \\ \check{V}_C \\ \check{W}_C \end{bmatrix} = \sum_i \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix}, \qquad \begin{bmatrix} \check{\Theta}_C \\ \check{\Phi}_C \\ \check{\Psi}_C \end{bmatrix} = \sum_i \begin{bmatrix} x_i - x_C \\ y_i - y_C \\ z_i - z_C \end{bmatrix} \wedge \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} \tag{195}
$$

$$\underbrace{\sum_i q_i \begin{bmatrix} \underline{\underline{\mathrm{I}}} \\ [\underline{\mathrm{x}}_i - \underline{\mathrm{x}}_C]_\wedge \end{bmatrix} \underline{\underline{\mathrm{S}}}_i}_{\underline{\underline{\mathrm{A}}}} \ \underline{\mathrm{a}} = \underbrace{\begin{bmatrix} \check{U}_C \\ \check{V}_C \\ \check{W}_C \\ \check{\Theta}_C \\ \check{\Phi}_C \\ \check{\Psi}_C \end{bmatrix}}_{\underline{\check{\mathrm{U}}}_C}, \tag{196}$$

$$\underline{\mathrm{a}} = \underline{\underline{\mathrm{A}}}^{-1} \underline{\check{\mathrm{U}}}_C \tag{197}$$

$$\underline{\mathrm{U_i}} = q_i \left(\underline{\underline{\mathrm{S}}}_i \underline{\underline{\mathrm{A}}}^{-1} \right) \underline{\check{\mathrm{U}}}_C \tag{198}$$

$$\underline{\underline{\mathrm{A}}}^{-1} = \underbrace{\begin{bmatrix} \frac{1}{m} \underline{\underline{\mathrm{I}}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\mathrm{J}}}^{-1} \end{bmatrix}}_{\underline{\underline{\mathrm{M}}}^{-1}} \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{I}}} & \underline{\underline{0}} \\ [\underline{\mathrm{x}}_C - \underline{\mathrm{x}}_G]_\wedge & \underline{\underline{\mathrm{I}}} \end{bmatrix}}_{\underline{\underline{\mathrm{L}}}_{CG}^\top}, \tag{199}$$

$$\underline{\mathrm{R}}_{abc,i} = q_i \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{q_i}{\sum_i q_i} \left\{ \check{U}_C, \check{V}_C, \check{W}_C \right\}; \tag{200}$$

$$\underline{\mathrm{R}}_{d,i} + \underline{\mathrm{R}}_{e,i} + \underline{\mathrm{R}}_{f,i} = q_i \left(d \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \wedge \underline{\mathrm{x}}_{Gi} \tag{201}$$

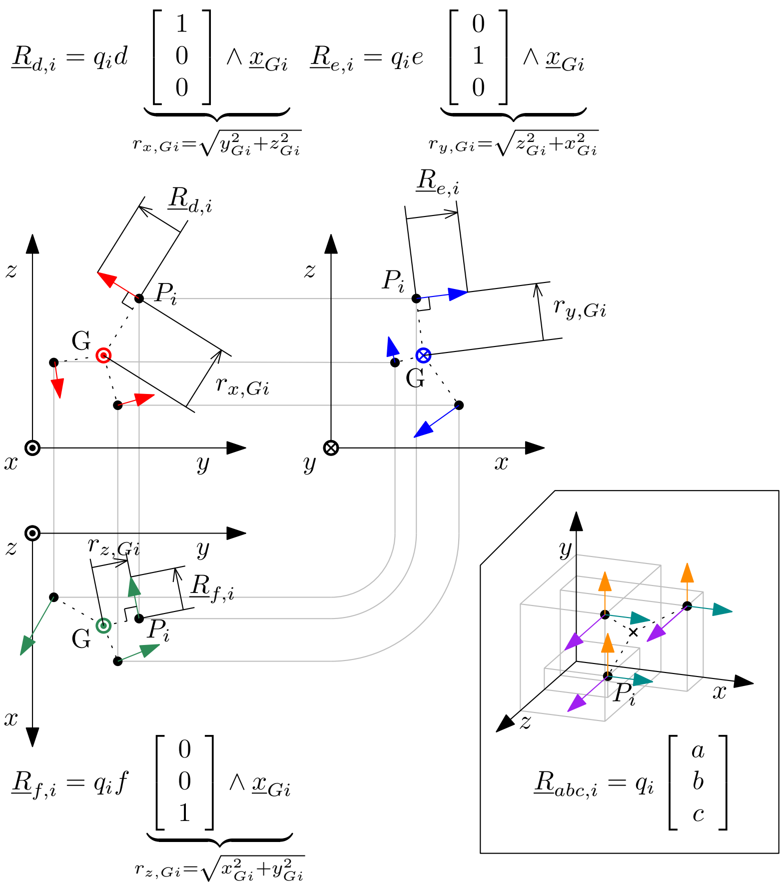
$$\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \underline{\underline{\mathrm{J}}}^{-1} \begin{bmatrix} \check{\Theta}_G \\ \check{\Phi}_G \\ \check{\Psi}_G \end{bmatrix} \tag{202}$$

$$0 = \underline{\delta \mathrm{u}}_C^\top \left(-\underline{\check{\mathrm{U}}}_C \right) + \sum_i \underline{\delta \mathrm{u}}_i^\top \underline{\mathrm{U}}_i \tag{203}$$

$$= \left(-\underline{\delta \mathrm{u}}_C^\top + \sum_i q_i \underline{\delta \mathrm{u}}_i^\top \underline{\underline{\mathrm{S}}}_i \underline{\underline{\mathrm{A}}}^{-1} \right) \underline{\check{\mathrm{U}}}_C \tag{204}$$

$$\underline{\delta \mathrm{u}}_C = \begin{bmatrix} \cdots & q_i \underline{\underline{\mathrm{A}}}^{-\top} \underline{\underline{\mathrm{S}}}_i^\top & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \underline{\delta \mathrm{u}}_i \\ \vdots \end{bmatrix}, \tag{205}$$

$$\underline{\delta \mathrm{u}}_C^\top = \begin{bmatrix} \delta u_C & \delta v_C & \delta w_C & \delta \theta_C & \delta \phi_C & \delta \psi_C \end{bmatrix}$$
$$\underline{\delta \mathrm{u}}_i^\top = \begin{bmatrix} \delta u_i & \delta v_i & \delta w_i \end{bmatrix}$$

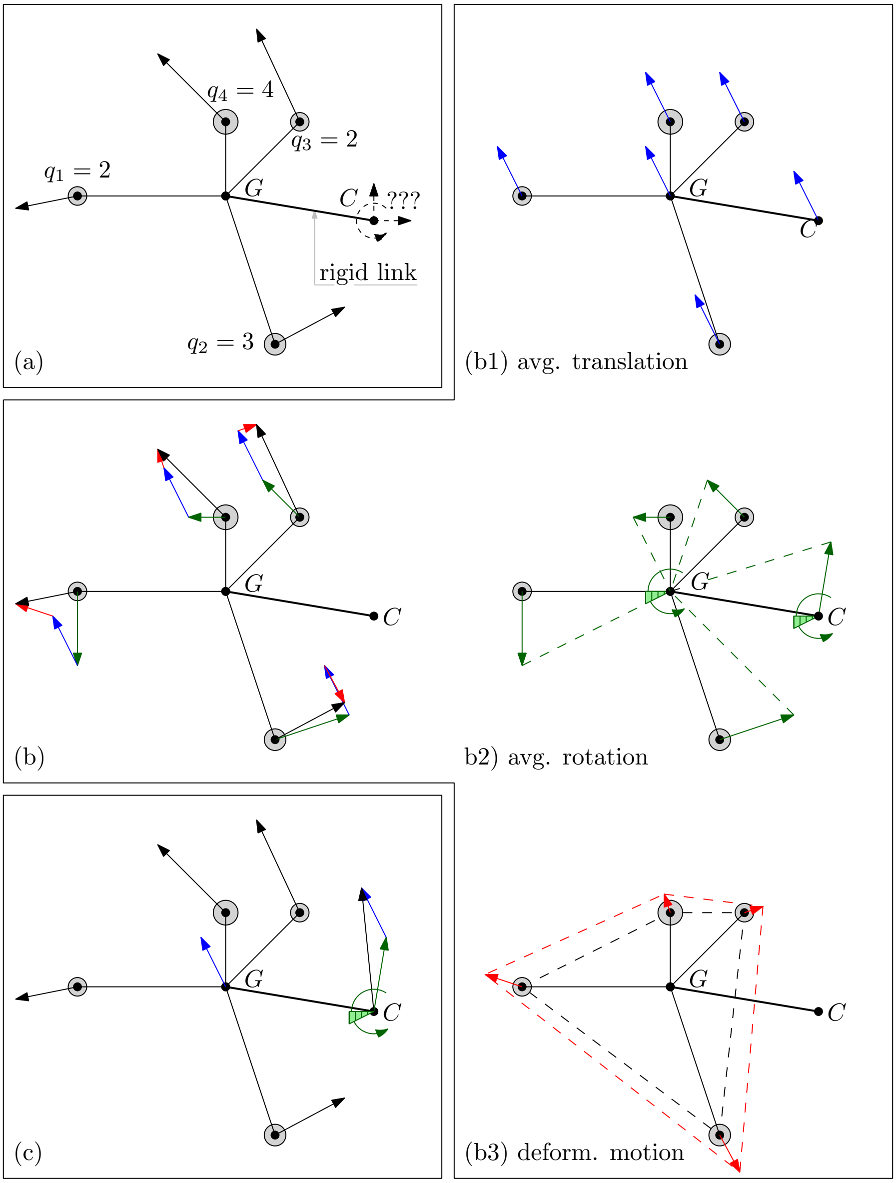


$\underline{R}_{d,i} = q_i d \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \wedge \underline{x}_{Gi}}_{r_{x,Gi}=\sqrt{y_{Gi}^2+z_{Gi}^2}}$
$\underline{R}_{e,i} = q_i e \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \wedge \underline{x}_{Gi}}_{r_{y,Gi}=\sqrt{z_{Gi}^2+x_{Gi}^2}}$
$\underline{R}_{d,i}$
$\underline{R}_{e,i}$
z
P_i
G
$r_{x,Gi}$
$r_{y,Gi}$
x
y
$r_{z,Gi}$
$\underline{R}_{f,i}$
$\underline{R}_{f,i} = q_i f \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \wedge \underline{x}_{Gi}}_{r_{z,Gi}=\sqrt{x_{Gi}^2+y_{Gi}^2}}$
$\underline{R}_{abc,i} = q_i \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\underline{u}_C = \sum_i q_i \underline{\underline{A}}^{-\top} \underline{\underline{S}}_i^{\top} \underline{u}_i = \cdots = \underbrace{\begin{bmatrix} \underline{\underline{I}} & -[\underline{x}_C - \underline{x}_G]_\wedge \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix}}_{\underline{\underline{L}}_{CG}} \underline{u}_G \qquad (206)$$

$$\begin{bmatrix} u_G \\ v_G \\ w_G \end{bmatrix} = \frac{1}{m} \sum_i q_i \underline{u}_i, \qquad \begin{bmatrix} \theta_G \\ \phi_G \\ \psi_G \end{bmatrix} = \underline{\underline{J}}^{-1} \underbrace{\sum_i q_i [\underline{x}_i - \underline{x}_G]_\wedge \underline{u}_i}_{P_i \text{ disps. moment}}.$$

$$\underline{0} = \frac{1}{m} \sum_i q_i \underline{u}_i, \qquad \underline{0} = \sum_i q_i [\underline{x}_i - \underline{x}_G]_\wedge \underline{u}_i$$


$q_4 = 4$
$q_3 = 2$
$q_1 = 2$
G
C
???
rigid link
$q_2 = 3$
(a)
(b1) avg. translation
(b)
b2) avg. rotation
(c)
(b3) deform. motion

$$\ddot{\underline{\mathrm{d}}} = \underbrace{\begin{bmatrix} \cdots & \underline{\mathrm{t}}_l & \cdots \end{bmatrix}}_{\underline{\underline{\mathrm{T}}}} \underbrace{\begin{bmatrix} \vdots \\ \alpha_l \\ \vdots \end{bmatrix}}_{\underline{\alpha}}, \tag{207}$$

$$\underline{\underline{\mathrm{T}}}^{\top} \underline{\underline{\mathrm{M}}}\, \underline{\underline{\mathrm{T}}}\, \underline{\alpha} = \underline{\underline{\mathrm{T}}}^{\top} \underline{\mathrm{F}} \tag{208}$$

$$\underline{\underline{\mathrm{M}}}\, \underline{\underline{\mathrm{T}}}\, \underline{\alpha} = \underline{\mathrm{F}}\ [+\underline{\mathrm{R}}_l]$$

$$\underline{\underline{\mathrm{K}}}\, \underline{\mathrm{d}} = \underline{\mathrm{F}} - \underline{\underline{\mathrm{M}}}\, \underline{\underline{\mathrm{T}}}\, \underline{\alpha}\,, \tag{209}$$

$$\underline{\underline{\mathrm{M}}}\,\ddot{\underline{\mathrm{d}}} + \underline{\underline{\mathrm{C}}}\,\dot{\underline{\mathrm{d}}} + \underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} = \underline{\mathrm{f}}\,(t), \quad \underline{\mathrm{d}} = \underline{\mathrm{d}}\,(t) \tag{210}$$

$$\underline{\underline{\mathrm{M}}}_{\mathrm{R}}\,\ddot{\underline{\mathrm{r}}} + \underline{\underline{\mathrm{C}}}_{\mathrm{R}}\,\dot{\underline{\mathrm{r}}} + \underline{\underline{\mathrm{K}}}_{\mathrm{R}}\,\underline{\mathrm{r}} = \underline{\mathrm{f}}_{\mathrm{R}}(t) + \underbrace{\underline{\underline{\Lambda}}^{\top}\,\underline{\mathrm{R}}\,(t)}_{=0} \tag{211}$$

$$\left\{\underline{\underline{\mathrm{M}}}_{\mathrm{R}}, \underline{\underline{\mathrm{C}}}_{\mathrm{R}}, \underline{\underline{\mathrm{K}}}_{\mathrm{R}}\right\} = \underline{\underline{\Lambda}}^{\top}\left\{\underline{\underline{\mathrm{M}}}, \underline{\underline{\mathrm{C}}}, \underline{\underline{\mathrm{K}}}\right\}\underline{\underline{\Lambda}}$$

$$\underline{\mathrm{f}}_{\mathrm{R}}(t) = \underline{\underline{\Lambda}}^{\top}\left(\underline{\mathrm{f}}\,(t) - \underline{\underline{\mathrm{M}}}\,\ddot{\underline{\Delta}} - \underline{\underline{\mathrm{C}}}\,\dot{\underline{\Delta}} - \underline{\underline{\mathrm{K}}}\,\underline{\Delta}\right)$$

$$\underline{\mathrm{f}}\,(t) = \frac{\bar{\underline{\mathrm{f}}}\,e^{j\omega t} + \bar{\underline{\mathrm{f}}}^{\,*}e^{-j\omega t}}{2} = \mathrm{Re}(\bar{\underline{\mathrm{f}}}\,e^{j\omega t}) \tag{212}$$

$$\underline{\mathrm{f}}\,(t) = \bar{\underline{\mathrm{f}}}\,e^{j\omega t} \tag{213}$$

$$\mathrm{Re}(\bar{\underline{\mathrm{f}}}\,e^{j\omega t}) = \mathrm{Re}(\bar{\underline{\mathrm{f}}})\cos\omega t - \mathrm{Im}(\bar{\underline{\mathrm{f}}})\sin\omega t \tag{214}$$

$$\underline{\mathrm{d}}\,(t) = \bar{\underline{\mathrm{d}}}\,e^{j\omega t} \tag{215}$$

$$\left(-\omega^2\,\underline{\underline{\mathrm{M}}} + j\omega\,\underline{\underline{\mathrm{C}}} + \underline{\underline{\mathrm{K}}}\right)\bar{\underline{\mathrm{d}}} = \bar{\underline{\mathrm{f}}} \tag{216}$$

$$\left(-\omega^2\,\underline{\underline{\mathrm{M}}} + \underline{\underline{\mathrm{K}}}\right)\bar{\underline{\mathrm{d}}} = \underline{0} \tag{217}$$

$$\left(\underline{\underline{\mathrm{M}}}^{-1}\,\underline{\underline{\mathrm{K}}} - \omega^2\,\underline{\underline{\mathrm{I}}}\right)\hat{\underline{\mathrm{d}}} = \underline{0}\,; \tag{218}$$

$$m_i = \hat{\underline{\mathrm{d}}}_i^{\top}\,\underline{\underline{\mathrm{M}}}\,\hat{\underline{\mathrm{d}}}_i = 1 \tag{219}$$

$$\underline{\mathrm{x}}\,(t) = a\,\hat{\underline{\mathrm{d}}}_i\sin(\omega_i t) \tag{220}$$

$$f(t) = \bar{\underline{\mathrm{f}}}\,\cos(\omega_i t), \tag{221}$$

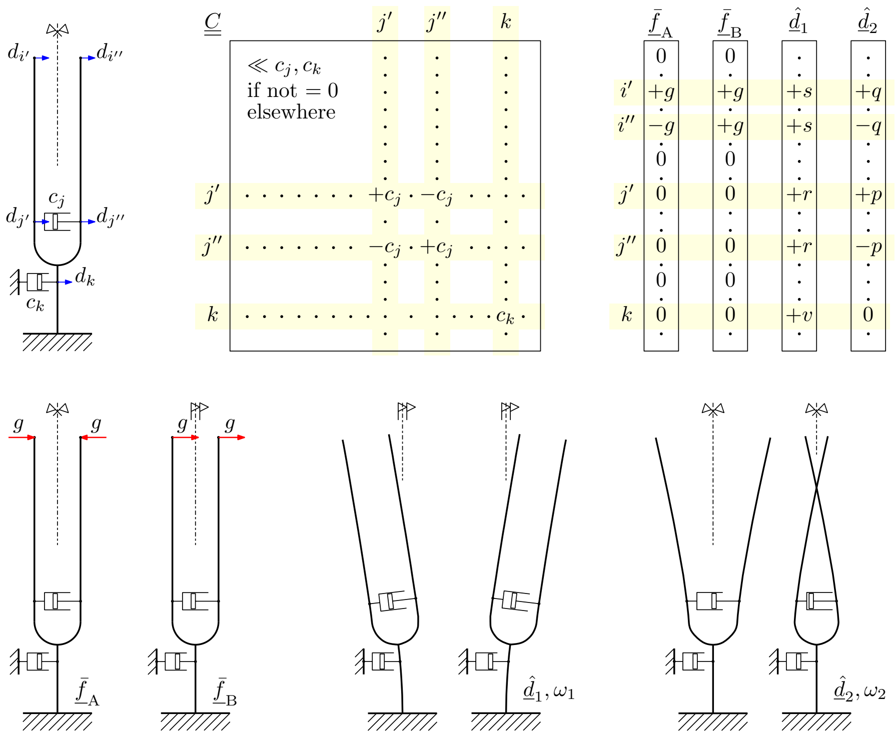
$$\underbrace{\left(-\omega_i^2\,\underline{\underline{\mathrm{M}}} + \underline{\underline{\mathrm{K}}}\right)\hat{\underline{\mathrm{d}}}_i}_{=\underline{0}}\,a_i\sin(\omega_i t) + \omega_i a_i\,\underline{\underline{\mathrm{C}}}\,\hat{\underline{\mathrm{d}}}_i\cos(\omega_i t) = \bar{\underline{\mathrm{f}}}\,\cos(\omega_i t). \tag{222}$$

$$a_i = \frac{\hat{\underline{\mathrm{d}}}^{\top}\,\bar{\underline{\mathrm{f}}}}{\omega_i\,\hat{\underline{\mathrm{d}}}^{\top}\,\underline{\underline{\mathrm{C}}}\,\hat{\underline{\mathrm{d}}}_i} \tag{223}$$

$$\hat{\underline{\mathrm{d}}}_j^{\top}\,\underline{\underline{\mathrm{M}}}\,\hat{\underline{\mathrm{d}}}_i = m_i\delta_{ij} \qquad \hat{\underline{\mathrm{d}}}_j^{\top}\,\underline{\underline{\mathrm{K}}}\,\hat{\underline{\mathrm{d}}}_i = m_i\omega_i^2\delta_{ij} \tag{224}$$

$$\underline{\underline{\Xi}} = \left[\hat{\underline{\mathrm{d}}}_1 \;\; \cdots \;\; \hat{\underline{\mathrm{d}}}_l \;\; \cdots \;\; \hat{\underline{\mathrm{d}}}_m\right], \tag{225}$$

$$\bar{\underline{\mathrm{d}}} = \underline{\underline{\Xi}}\,\bar{\underline{\xi}} \tag{226}$$



applied loads

$\underline{f}_A(t) = \mathrm{Re}\left(\bar{\underline{f}}_A e^{j\omega t}\right)$

$\underline{f}_B(t) = \mathrm{Re}\left(\bar{\underline{f}}_B e^{j\omega t}\right)$

$\left\langle \hat{\underline{d}}_1, \underline{f}_A \right\rangle = 0$

$\left\langle \hat{\underline{d}}_1, \underline{f}_B \right\rangle = 2gs > 0$

$\left\langle \hat{\underline{d}}_1, \underline{\underline{C}}\, \hat{\underline{d}}_1 \right\rangle = c_k v^2 + \varepsilon$

$\left\langle \hat{\underline{d}}_2, \underline{f}_A \right\rangle = 2gq > 0$

$\left\langle \hat{\underline{d}}_2, \underline{f}_B \right\rangle = 0$

$\left\langle \hat{\underline{d}}_2, \underline{\underline{C}}\, \hat{\underline{d}}_2 \right\rangle = 4c_j p^2 + \varepsilon$

.

$$\underline{\underline{\Xi}}^\top \underline{\underline{\mathrm{M}}}\,\underline{\underline{\Xi}} = \underline{\underline{\mathrm{I}}} \qquad \underline{\underline{\Xi}}^\top \underline{\underline{\mathrm{K}}}\,\underline{\underline{\Xi}} = \underline{\underline{\Omega}} = \mathrm{diag}(\omega_l^2); \tag{227}$$

$$\underline{\underline{\mathrm{C}}} = \alpha\,\underline{\underline{\mathrm{M}}} + \beta\,\underline{\underline{\mathrm{K}}} \tag{228}$$

$$\underline{\underline{\Xi}}^\top \left(-\omega^2\,\underline{\underline{\mathrm{M}}} + j\omega\,\underline{\underline{\mathrm{C}}} + \underline{\underline{\mathrm{K}}}\right) \underline{\underline{\Xi}}\,\underline{\bar{\xi}} = \underline{\underline{\Xi}}^\top \underline{\bar{\mathrm{f}}} \tag{229}$$

$$\left(-\omega^2\,\underline{\underline{\mathrm{I}}} + j\omega\left(\alpha\,\underline{\underline{\mathrm{I}}} + \beta\,\underline{\underline{\Omega}}\right) + \underline{\underline{\Omega}}\right) \underline{\bar{\xi}} = \underline{\underline{\Xi}}^\top \underline{\bar{\mathrm{f}}}\,, \tag{230}$$

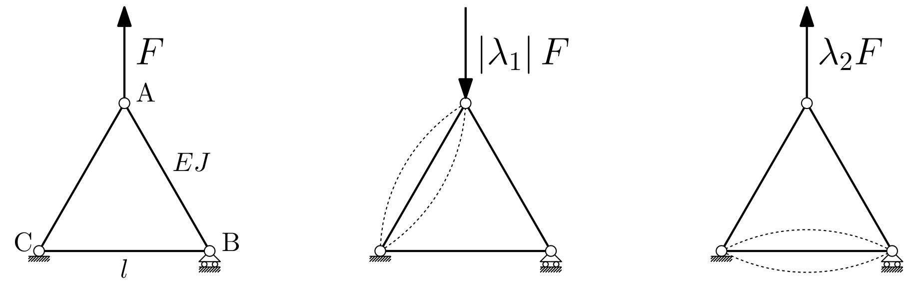
$$\left(-\omega^2 + j\omega\left(\alpha + \beta\omega_l^2\right) + \omega_l^2\right) \xi_l = q_l, \quad j = 1 \dots m \tag{231}$$

$$\begin{aligned} \xi_l(t) &= \mathrm{Re}(\bar{\xi}_l)\cos\omega t - \mathrm{Im}(\bar{\xi}_l)\sin\omega t \\ &= \left|\bar{\xi}_l\right| \cos\left(\omega t + \psi_l - \phi_l\right) \end{aligned}$$

$$a_l = 1 - r_l^2 \qquad b_l = 2\zeta_l r_l \qquad r_l = \frac{\omega}{\omega_l}$$

$$\begin{aligned} \left|\bar{\xi}_l\right| &= \frac{|\bar{q}_l|}{\omega_l^2} \frac{1}{\sqrt{a_l^2 + b_l^2}} \\ \psi_l &= \arg(\bar{q}_l) \\ \phi_l &= \arg(a_l + j b_l) \end{aligned}$$

$$\begin{aligned} \mathrm{Re}(\bar{\xi}_l) &= \frac{1}{\omega_l^2} \frac{a_l\,\mathrm{Re}(\bar{q}_l) + b_l\,\mathrm{Im}(\bar{q}_l)}{a_l^2 + b_l^2} \\ \mathrm{Im}(\bar{\xi}_l) &= \frac{1}{\omega_l^2} \frac{a_l\,\mathrm{Im}(\bar{q}_l) - b_l\,\mathrm{Re}(\bar{q}_l)}{a_l^2 + b_l^2}. \end{aligned}$$



$$N_{AB} = N_{CA} = +\frac{F}{\sqrt{3}} \qquad \lambda_1 = -\frac{\pi^2 EJ}{l^2}\frac{\sqrt{3}}{F} \qquad \lambda_2 = +\frac{\pi^2 EJ}{l^2}\frac{2\sqrt{3}}{F}$$

$$N_{BC} = -\frac{F}{2\sqrt{3}} \qquad \text{moltepl. } 2$$

$$N_{\mathrm{crit},i} = -\frac{\pi^2 i^2 EJ}{l^2} \qquad \text{minimum } \lambda_i \text{ in absolute value} \qquad \text{minimum } \lambda_i > 0$$

$$\begin{aligned}
\delta U_{\mathrm{i}} &= \iiint_V \delta\,\underline{\epsilon}^{\top}\left(\lambda\,\underline{\sigma}_{\,0} + \underline{\underline{\mathrm{D}}}\;\underline{\epsilon}\right) dV \\
&= \iiint_V \left[\,\underline{\underline{\mathrm{B}}}\,(\,\underline{\mathrm{d}}\,)\delta\,\underline{\mathrm{d}}\,\right]^{\top}\left(\lambda\,\underline{\sigma}_{\,0} + \underline{\underline{\mathrm{D}}}\;\underline{\underline{\mathrm{B}}}\,(\,\underline{\mathrm{d}}\,)\,\underline{\mathrm{d}}\right) dV \\
&= \dots \\
&= \delta\,\underline{\mathrm{d}}\;\left(\left(\underline{\underline{\mathrm{K}}}{}^{\mathrm{M}}_{\mathrm{ej}} + \lambda\,\underline{\underline{\mathrm{K}}}{}^{\mathrm{G}}_{\mathrm{ej}}\right)\,\underline{\mathrm{d}} + \mathrm{o}\,(\,\underline{\mathrm{d}}\,)\right).
\end{aligned}$$

$$\left(\underline{\underline{\mathrm{K}}}^{\,\mathrm{M}} + \lambda\,\underline{\underline{\mathrm{K}}}^{\,\mathrm{G}}\right)\,\underline{\delta\mathrm{d}} \;=\; \underline{\delta\mathrm{F}} \tag{232}$$

$$\left(\underline{\underline{\mathrm{K}}}^{\,\mathrm{M}} + \lambda_i\,\underline{\underline{\mathrm{K}}}^{\,\mathrm{G}}\right)\,\underline{\delta\hat{\mathrm{d}}}_{\,i} = 0 \tag{233}$$

$$\underline{\mathrm{r}}\left(\underline{\mathrm{u}}\right) = \underline{0} \tag{234}$$

$$\underline{\mathrm{r}}\left(\underline{\mathrm{u}}\right) = \underline{\mathrm{G}}\left(\underline{\mathrm{u}}\right) - \underline{\mathrm{f}}\left(\underline{\mathrm{u}}\right)$$

$$\underline{\mathrm{r}}\left(\underline{\mathrm{u}}^{*}\right) = \underline{\mathrm{r}}\left(\underline{\mathrm{u}}^{i}\right) + \underline{\underline{\mathrm{J}}}_{r}\left(\underline{\mathrm{u}}^{\mathrm{i}}\right) \cdot \left(\underline{\mathrm{u}}^{*} - \underline{\mathrm{u}}^{i}\right) + o\left(\underline{\mathrm{u}}^{*} - \underline{\mathrm{u}}^{i}\right) = \underline{0}\,. \tag{235}$$

$$\left[\underline{\underline{\mathrm{J}}}_{r}\left(\underline{\mathrm{u}}^{i}\right)\right]_{l,m} = \left[\underline{\underline{\mathrm{J}}}_{r}^{i}\right]_{l,m} = \left.\frac{\partial r_l}{\partial u_m}\right|_{\underline{\mathrm{u}}=\underline{\mathrm{u}}^{i}}, \quad l,m = 1\dots n \tag{236}$$

$$\underline{\underline{\mathrm{J}}}_{r}^{i} = \underline{\underline{\mathrm{J}}}_{G}^{i} - \underline{\underline{\mathrm{J}}}_{f}^{i} \tag{237}$$

$$\left[\underline{\underline{\mathrm{J}}}_{G}^{i}\right]_{l,m} = \left.\frac{\partial G_l}{\partial u_m}\right|_{\underline{\mathrm{u}}=\underline{\mathrm{u}}^{i}}, \quad \left[\underline{\underline{\mathrm{J}}}_{f}^{i}\right]_{l,m} = \left.\frac{\partial f_l}{\partial u_m}\right|_{\underline{\mathrm{u}}=\underline{\mathrm{u}}^{i}}, \quad l,m = 1\dots n \tag{238}$$

$$\underbrace{\underline{\underline{\mathrm{J}}}_{r}^{i}}_{\underline{\underline{\mathrm{K}}}^{i}}\underbrace{\left(\underline{\mathrm{u}}^{i+1} - \underline{\mathrm{u}}^{i}\right)}_{\underline{\Delta\mathrm{u}}^{i\to i+1}} = \underbrace{-\,\underline{\mathrm{r}}\left(\underline{\mathrm{u}}^{i}\right)}_{\underline{\Delta\mathrm{r}}^{i\to i+1}} \tag{239}$$

$$\underline{\mathrm{u}}^{i+1} = \underline{\mathrm{u}}^{i} - \underline{\underline{\mathrm{J}}}_{r}^{i}\backslash\underline{\mathrm{r}}\left(\underline{\mathrm{u}}^{i}\right) \tag{240}$$

Bibliography

[1] A. E. H. Love, *A treatise on the mathematical theory of elasticity.* Cambridge university press, 2013.

[2] S. Timoshenko and J. Goodier, *Theory of Elasticity.* McGraw-Hill book Company, 1951.

[3] S. Vlachoutsis, "Shear correction factors for plates and shells," *International Journal for Numerical Methods in Engineering*, vol. 33, no. 7, pp. 1537–1552, 1992.

[4] T. Chow, "On the propagation of flexural waves in an orthotropic laminated plate and its response to an impulsive load," *Journal of Composite Materials*, vol. 5, no. 3, pp. 306–319, 1971.

[5] V. Birman and C. W. Bert, "On the choice of shear correction factor in sandwich structures," *Journal of Sandwich Structures & Materials*, vol. 4, no. 1, pp. 83–95, 2002.

[6] C. Hua, "An inverse transformation for quadrilateral isoparametric elements: analysis and application," *Finite elements in analysis and design*, vol. 7, no. 2, pp. 159–166, 1990.

[7] B. M. Irons and A. Razzaque, "Experience with the patch test for convergence of finite elements," in *The mathematical foundations of the finite element method with applications to partial differential equations*, pp. 557–587, Elsevier, 1972.

[8] C. Militello and C. A. Felippa, "The individual element test revisited," in *The finite element method in the 1990's*, pp. 554–564, Springer, 1991.

[9] R. MacNeal, *Finite Elements.* CRC Press, 1993.

[10] T. J. Hughes and T. Tezduyar, "Finite elements based upon Mindlin plate theory with particular reference to the four-node bilinear isoparametric element," *Journal of applied mechanics*, vol. 48, no. 3, pp. 587–596, 1981.

[11] B. W. Lewis and M. Kane, "Svd subset selectionn," 2014.

[12] G. H. Golub and C. F. Van Loan, *Matrix computations.* JHU press, 2013.

[13] J. Oden, "Calculation of geometric stiffness matrices for complex structures.," *AIAA Journal*, vol. 4, no. 8, pp. 1480–1482, 1966.