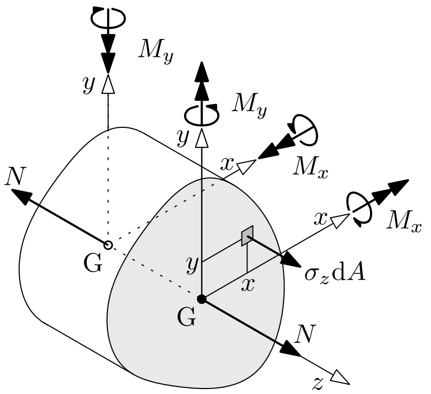
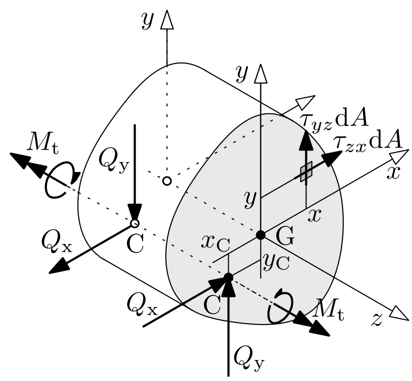
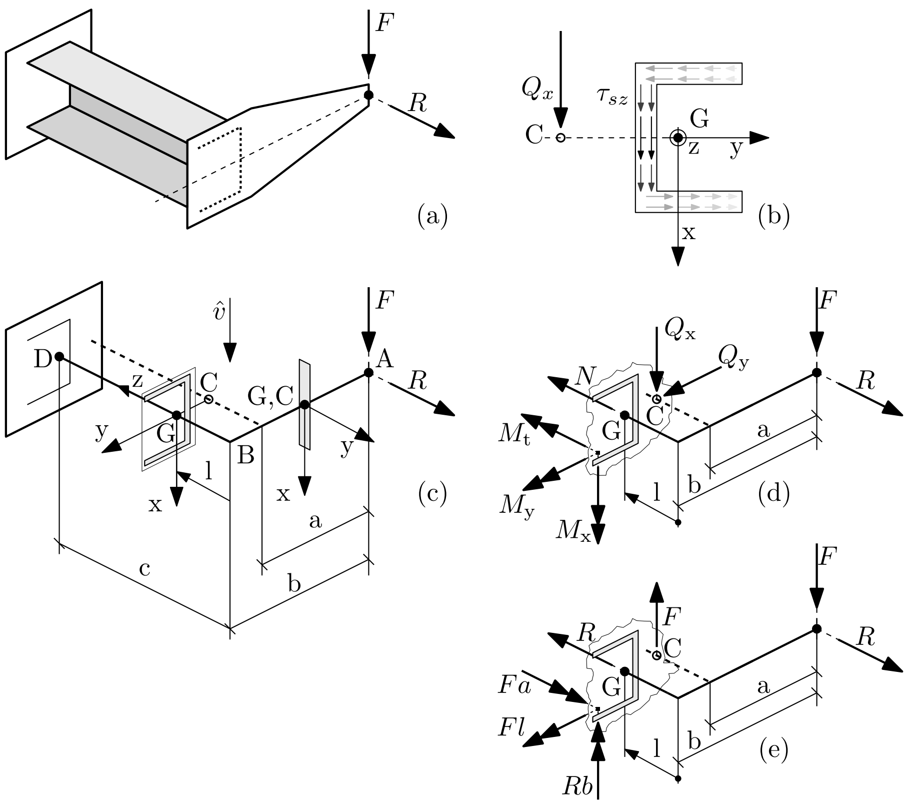




$$N = \int_{\mathcal{A}} \sigma_{zz} d\mathcal{A}$$
$$Q_y = \int_{\mathcal{A}} \tau_{yz} d\mathcal{A}$$
$$Q_x = \int_{\mathcal{A}} \tau_{zx} d\mathcal{A}$$

$$M_x \equiv M_{(G,x)} = \int_{\mathcal{A}} \sigma_z y d\mathcal{A}$$
$$M_y \equiv M_{(G,y)} = -\int_{\mathcal{A}} \sigma_z x d\mathcal{A}$$
$$M_t \equiv M_{(C,z)} = \int_{\mathcal{A}} \left[\tau_{yz}(x - x_C) - \tau_{zx}(y - y_C)\right] d\mathcal{A}$$


F
R
(a)
Q_x
τ_{sz}
C
G
z
y
x
(b)
$\hat{v}$
F
A
R
D
z
C
G,C
y
G
B
l
x
a
c
b
(c)
F
Q_x
Q_y
N
R
M_t
G
C
a
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Rb
(e)

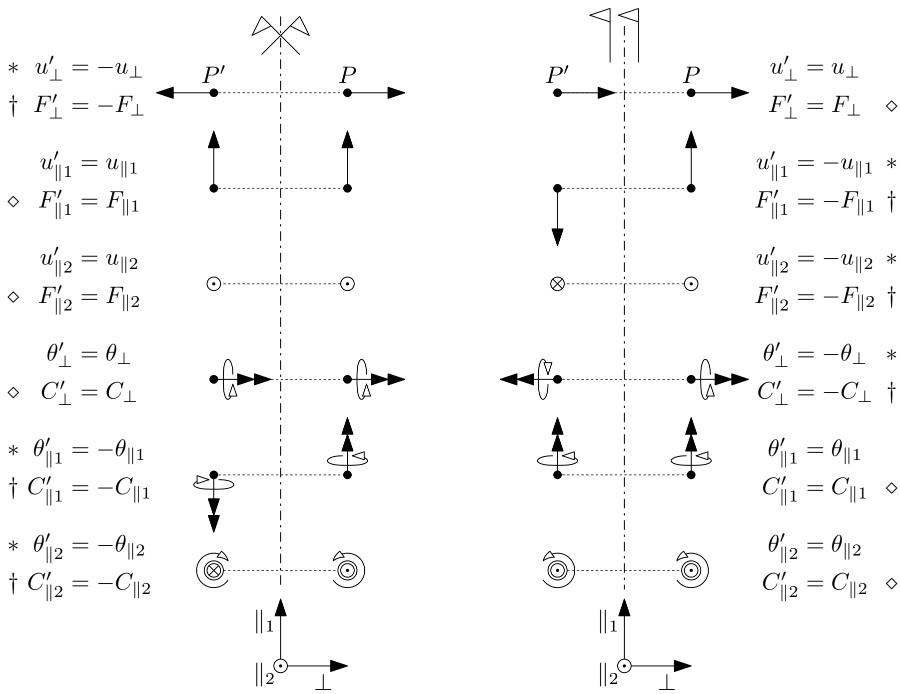
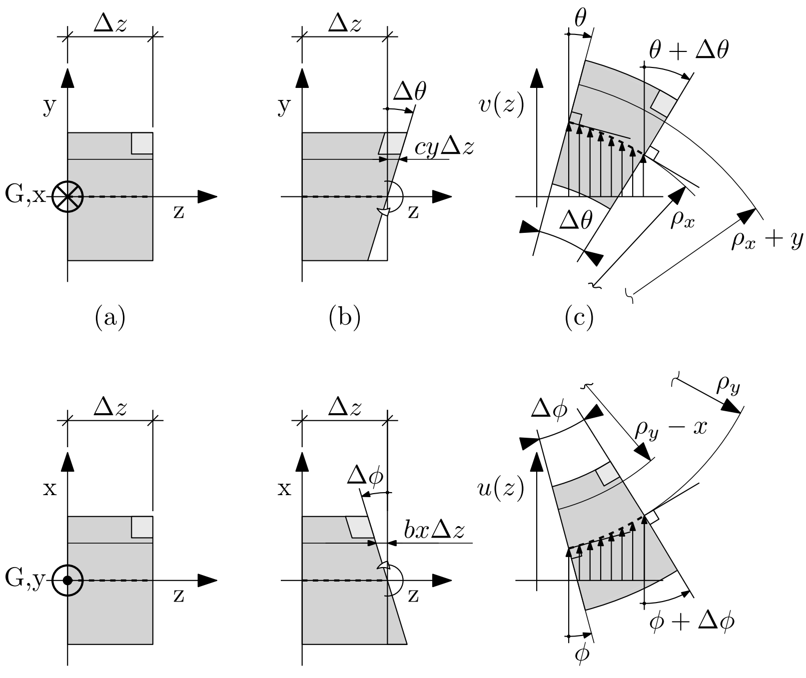


Figure 1: An overview of symmetrical and skew-symmetrical (generalized) loading and displacements.



(a) (b) (c) (d) (e) (f)

$$\epsilon_z = a + bx + cy \tag{1}$$

$$\frac{d\theta}{dz} = \frac{1}{\rho_x}, \quad \theta = -\frac{dv}{dz}, \quad \frac{d^2v}{dz^2} = -\frac{1}{\rho_x} \tag{2}$$

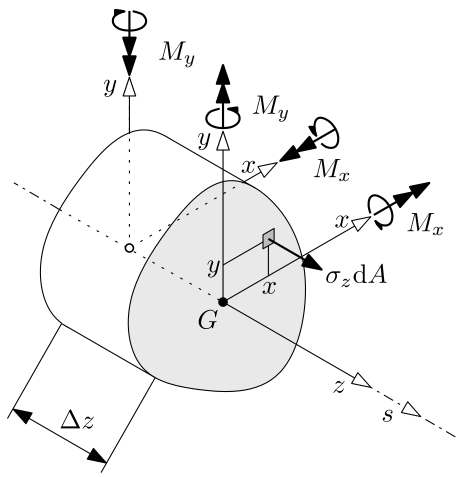
$$\frac{d\phi}{dz} = \frac{1}{\rho_y}, \quad \phi = +\frac{du}{dz}, \quad \frac{d^2u}{dz^2} = +\frac{1}{\rho_y} \tag{3}$$

$$\sigma_z = E_z\epsilon_z; \quad \epsilon_z = e - \frac{1}{\rho_y}x + \frac{1}{\rho_x}y \tag{4}$$

$$N = \iint_{\mathcal{A}} E_z\epsilon_z dA = \overline{EA}e \tag{5}$$

$$M_x = \iint_{\mathcal{A}} E_z\epsilon_z y dA = \overline{EJ}_{xx}\frac{1}{\rho_x} - \overline{EJ}_{xy}\frac{1}{\rho_y} \tag{6}$$

$$M_y = -\iint_{\mathcal{A}} E_z\epsilon_z x dA = -\overline{EJ}_{xy}\frac{1}{\rho_x} + \overline{EJ}_{yy}\frac{1}{\rho_y} \tag{7}$$



$$\overline{EA} = \iint_{\mathcal{A}} E_z(x,y)\ dA \tag{8}$$

$$\overline{EJ}_{xx} = \iint_{\mathcal{A}} E_z(x,y) yy\ dA \tag{9}$$

$$\overline{EJ}_{xy} = \iint_{\mathcal{A}} E_z(x,y) yx\ dA \tag{10}$$

$$\overline{EJ}_{yy} = \iint_{\mathcal{A}} E_z(x,y) xx\ dA \tag{11}$$

$$e = \frac{N}{\overline{\overline{EA}}}. \tag{12}$$

$$\frac{1}{\rho_x} = \frac{M_x \overline{EJ}_{yy} + M_y \overline{EJ}_{xy}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{13}$$

$$\frac{1}{\rho_y} = \frac{M_x \overline{EJ}_{xy} + M_y \overline{EJ}_{xx}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{14}$$

$$\frac{1}{\rho_{\mathrm{eq}}} = \sqrt{\frac{1}{\rho_x^2} + \frac{1}{\rho_y^2}} \tag{15}$$

$$\sigma_z = E_z \epsilon_z; \quad \epsilon_z = \alpha M_x + \beta M_y + \gamma N \tag{16}$$

$$\alpha\left(x, y, \overline{EJ}_{**}\right) = \frac{-\overline{EJ}_{xy}x + \overline{EJ}_{yy}y}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{17}$$

$$\beta\left(x, y, \overline{EJ}_{**}\right) = \frac{-\overline{EJ}_{xx}x + \overline{EJ}_{xy}y}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{18}$$

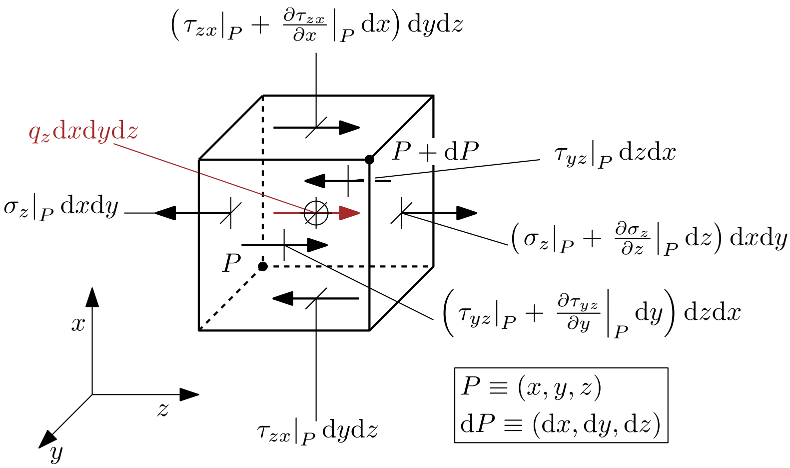
$$\gamma\left(\overline{EA}\right) = \frac{1}{\overline{EA}}. \tag{19}$$

$$(x_N, y_N) \equiv e\rho_{\mathrm{eq}}^2 \left(\frac{1}{\rho_y}, -\frac{1}{\rho_x}\right);$$

$$\hat{n}_\parallel = \rho_{\mathrm{eq}} \left(\frac{1}{\rho_x}, \frac{1}{\rho_y}\right),$$

$$\hat{n}_\perp = \rho_{\mathrm{eq}} \left(-\frac{1}{\rho_y}, \frac{1}{\rho_x}\right),$$

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \zeta \hat{n}_\parallel = \lambda \begin{bmatrix} \frac{1}{\rho_x} \\ \frac{1}{\rho_y} \end{bmatrix} \tag{20}$$



$$S_y = \frac{dM_x}{dz}, \quad S_x = -\frac{dM_y}{dz}, \tag{21}$$

$$\frac{d\sigma_z}{dz} = E_z \alpha \left(x, y, \overline{EJ}_{**}\right) S_y - E_z \beta \left(x, y, \overline{EJ}_{**}\right) S_x \tag{22}$$

$$\frac{d\tau_{zx}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + q_z = 0 \tag{23}$$

$$\bar{\tau}_{zi} t = \int_{A^*} \frac{d\sigma_z}{dz} dA, \tag{24}$$

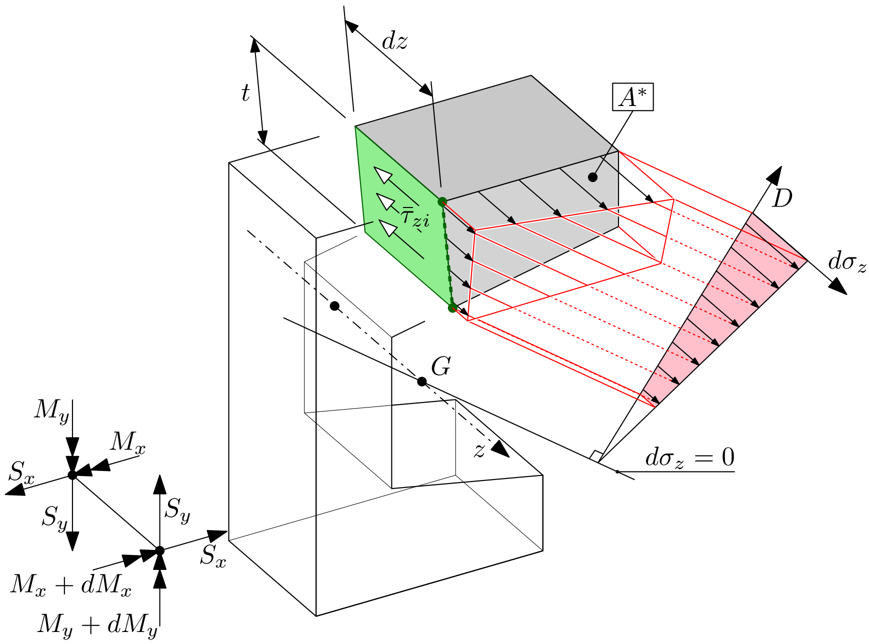
$$\bar{\tau}_{zi} = \frac{1}{t} \int_t \tau_{zi} dr \tag{25}$$

$$\bar{\tau}_{zi} t = \int_{A^*} \left(\frac{y S_y}{J_{xx}} + \frac{x S_x}{J_{yy}} \right) dA = \frac{\bar{y}^* A^*}{J_{xx}} S_y + \frac{\bar{x}^* A^*}{J_{yy}} S_x, \tag{26}$$

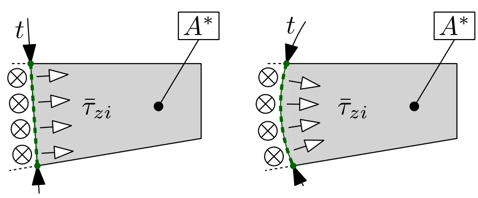
$$\bar{\tau}_{zi} t = q_{zi} = \int_0^s \int_{-t/2}^{t/2} \frac{d\sigma_z}{dz} dr d\varsigma \approx \int_0^s \left. \frac{d\sigma_z}{dz} \right|_{r=0} t d\varsigma. \tag{27}$$

$$\bar{\tau}_{zi}(s) t(s) = q(s) = \int_a^s \frac{d\sigma_z}{dz} t d\varsigma + \underbrace{\bar{\tau}_{zi}(a) t(a)}_{q_A}. \tag{28}$$

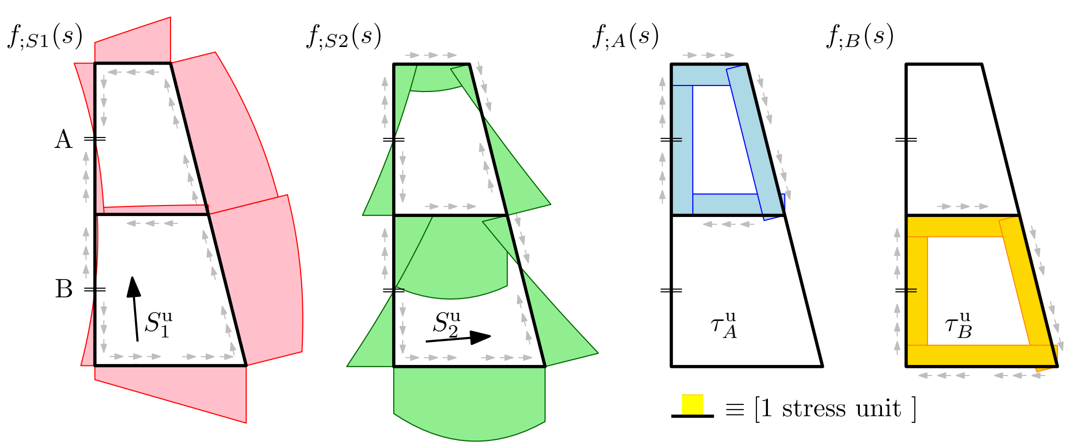
$$\tau(s) = \frac{S_1}{\mathcal{A}} f_{;S1}(s) + \frac{S_2}{\mathcal{A}} f_{;S2}(s) + \tau_A f_{;A}(s) + \tau_B f_{;B}(s) \tag{29}$$



dz
t
A^*
D
$\bar{\tau}_{zi}$
$d\sigma_z$
G
z
$d\sigma_z = 0$
M_y
M_x
S_x
S_y
S_y
S_x
$M_x + dM_x$
$M_y + dM_y$



t
A^*
$\bar{\tau}_{zi}$
t
A^*
$\bar{\tau}_{zi}$



$f_{;S1}(s)$
$f_{;S2}(s)$
$f_{;A}(s)$
$f_{;B}(s)$
A
B
S_1^{u}
S_2^{u}
τ_A^{u}
τ_B^{u}
≡ [1 stress unit]

(a) (b) (c) (d)

$$\Delta U = \int_s \frac{\tau^2}{2G_{sz}} t \Delta z ds \tag{30}$$

$$\frac{\partial \Delta U}{\partial \bar{\tau}_i} = \bar{\delta}_i t \Delta z \tag{31}$$

$$K_t = \frac{4A^2}{\oint \frac{1}{t} dl} \tag{32}$$

$$\tau_{\max} = \frac{M_t}{2t_{\min} A} \tag{33}$$

.

$$K_T \approx \frac{1}{3} \int_0^l t^3(s) ds \tag{34}$$

$$K_T \approx \frac{1}{3} \sum_i l_i t_i^3 \tag{35}$$

$$\tau_{\max} = \frac{M_t t_{\max}}{K_T} \tag{36}$$

$$q_i = \frac{\partial U}{\partial Q_i}$$

$$\frac{dU}{dl} = \frac{1}{2} \begin{pmatrix} N \\ M_\mathrm{x} \\ M_\mathrm{y} \\ Q_\mathrm{x} \\ Q_\mathrm{y} \\ M_\mathrm{t} \end{pmatrix}^\top \begin{pmatrix} a_{1,1} & g_{1,2} & g_{1,3} & i_{1,4} & i_{1,5} & i_{1,6} \\ 0 & b_{2,2} & e_{2,3} & i_{2,4} & i_{2,5} & i_{2,6} \\ 0 & 0 & b_{3,3} & i_{3,4} & i_{3,5} & i_{3,6} \\ 0 & 0 & 0 & c_{4,4} & f_{4,5} & h_{4,6} \\ 0 & 0 & 0 & 0 & c_{5,5} & h_{5,6} \\ 0 & 0 & 0 & 0 & 0 & d_{6,6} \end{pmatrix}_\mathrm{Sym} \begin{pmatrix} N \\ M_\mathrm{x} \\ M_\mathrm{y} \\ Q_\mathrm{x} \\ Q_\mathrm{y} \\ M_\mathrm{t} \end{pmatrix}, \tag{37}$$

$$\begin{aligned} a_{1,1} =& \frac{1}{EA} & \{b_{2,2}, b_{3,3}, e_{2,3}\} =& \frac{\{J_\mathrm{yy}, J_\mathrm{xx}, 2J_\mathrm{xy}\}}{E\left(J_\mathrm{xx}J_\mathrm{yy} - J_\mathrm{xy}^2\right)} \\ d_{6,6} =& \frac{1}{GK_t} & \{c_{4,4}, c_{5,5}, f_{4,5}\} =& \frac{\{\chi_\mathrm{x}, \chi_\mathrm{y}, \chi_\mathrm{xy}\},}{GA} \end{aligned}$$

$$
\begin{aligned}
u_P &= u + z\,(1+\check{\epsilon}_z)\,\frac{\cos\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}}\sin\phi \\
v_P &= v - z\,(1+\check{\epsilon}_z)\,\frac{\cos\phi}{\sqrt{1-\sin^2\phi\sin^2\theta}}\sin\theta \\
w_P &= w + z\left((1+\check{\epsilon}_z)\,\frac{\cos\phi\cos\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}} - 1\right),
\end{aligned}
$$

$$
\begin{aligned}
\check{\epsilon}_z(z) &= \frac{1}{z}\int_0^z \epsilon_z d\varsigma \\
&= \frac{1}{z}\int_0^z -\frac{\nu}{1-\nu}\,(\epsilon_x+\epsilon_y)\,d\varsigma,
\end{aligned}
$$

$$
u_P = u + z\phi \tag{38}
$$

$$
v_P = v - z\theta \tag{39}
$$

$$
w_P = w. \tag{40}
$$

$$
\frac{\partial w}{\partial x} = \bar{\gamma}_{zx} - \phi \tag{41}
$$

$$
\frac{\partial w}{\partial y} = \bar{\gamma}_{yz} + \theta \tag{42}
$$

$$
\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \tag{43}
$$

$$
\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial \theta}{\partial y} \tag{44}
$$

$$
\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{45}
$$

$$
= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x}\right) \tag{46}
$$

$$
\underline{\mathrm{e}} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} e_x \\ e_y \\ g_{xy} \end{bmatrix} \equiv \underline{\epsilon}_{\mathrm{Q}} \tag{47}
$$

$$\underline{\kappa} = \begin{bmatrix} +\frac{\partial \phi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x} \end{bmatrix} = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{48}$$

$$\underline{\epsilon}_{\mathrm{P}} \equiv \underline{\epsilon} = \underline{\mathrm{e}} + z\,\underline{\kappa}\,. \tag{49}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underline{\sigma} = \underline{\underline{\mathrm{D}}}\,\underline{\epsilon} = \underline{\underline{\mathrm{D}}}\,\underline{\mathrm{e}} + z\,\underline{\underline{\mathrm{D}}}\,\underline{\kappa}\,, \tag{50}$$

$$\underline{\underline{\mathrm{D}}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \tag{51}$$

$$\epsilon_z = -\frac{\nu}{1-\nu}\,(\epsilon_x + \epsilon_y)\,. \tag{52}$$

$$\begin{aligned} \underline{\mathrm{q}} = \begin{bmatrix} q_x \\ q_y \\ q_{xy} \end{bmatrix} &= \int_h \underline{\sigma}\,dz \\ &= \underbrace{\int_h \underline{\underline{\mathrm{D}}}\,dz}_{\underline{\underline{\mathrm{a}}}}\,\underline{\mathrm{e}} + \underbrace{\int_h \underline{\underline{\mathrm{D}}}\,zdz}_{\underline{\underline{\mathrm{b}}}}\,\underline{\kappa} \end{aligned} \tag{53}$$

$$\underline{\mathrm{q}}_{\,z} = \begin{bmatrix} q_{xz} \\ q_{yz} \end{bmatrix} \qquad q_{xz} = \int_h \tau_{zx} dz \qquad q_{yz} = \int_h \tau_{yz} dz. \tag{54}$$

$$\begin{aligned} \underline{\mathrm{m}} = \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} &= \int_h \underline{\sigma}\,zdz \\ &= \underbrace{\int_h \underline{\underline{\mathrm{D}}}\,zdz}_{\underline{\underline{\mathrm{b}}} \equiv \underline{\underline{\mathrm{b}}}^{\mathrm{T}}}\,\underline{\mathrm{e}} + \underbrace{\int_h \underline{\underline{\mathrm{D}}}\,z^2 dz}_{\underline{\underline{\mathrm{c}}}}\,\underline{\kappa}\,. \end{aligned} \tag{55}$$

$$\begin{bmatrix} \underline{\mathrm{q}} \\ \underline{\mathrm{m}} \end{bmatrix} = \begin{bmatrix} \underline{\underline{\mathrm{a}}} & \underline{\underline{\mathrm{b}}} \\ \underline{\underline{\mathrm{b}}}^{\mathrm{T}} & \underline{\underline{\mathrm{c}}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{e}} \\ \underline{\kappa} \end{bmatrix} \tag{56}$$

$$\upsilon^{\dagger} = \frac{1}{2}\left[\begin{array}{c} \underline{q} \\ \underline{m} \end{array}\right]^{\top}\left[\begin{array}{c} \underline{e} \\ \underline{\kappa} \end{array}\right] \tag{57}$$

$$= \frac{1}{2}\left[\begin{array}{c} \underline{e} \\ \underline{\kappa} \end{array}\right]^{\top}\left[\begin{array}{cc} \underline{\underline{a}} & \underline{\underline{b}} \\ \underline{\underline{b}}^{\mathrm{T}} & \underline{\underline{c}} \end{array}\right]\left[\begin{array}{c} \underline{e} \\ \underline{\kappa} \end{array}\right]. \tag{58}$$

$$\underline{\underline{a}} = h\,\underline{\underline{D}} \qquad \underline{\underline{b}} = \underline{\underline{0}} \qquad \underline{\underline{c}} = \frac{h^3}{12}\underline{\underline{D}},$$

$$\underline{\gamma}_z = \left[\begin{array}{c} \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \end{array}\right]$$

$$\underline{q}_z = \left[\begin{array}{c} q_{xz} \\ q_{yz} \end{array}\right]$$

$$\upsilon^{\ddagger} = \frac{1}{2}\underline{\gamma}_z^{\top}\underline{q}_z = \frac{1}{2}\bar{\gamma}_{xz}q_{xz} + \frac{1}{2}\bar{\gamma}_{yz}q_{yz}. \tag{59}$$

$$\upsilon^{\ddagger} = \frac{1}{2}\underline{\gamma}_z^{\top}\underbrace{\left[\chi\left(\frac{1}{h}\int_h \underline{\underline{G}}^{-1}dz\right)^{-1}h\right]}_{\underline{\underline{\Gamma}}}\underline{\gamma}_z \tag{60}$$

$$\underline{\underline{G}} = \frac{E}{2\,(1+\nu)}\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right],$$

$$\underline{q}_z = \underline{\underline{\Gamma}}\,\underline{\gamma}_z. \tag{61}$$

$$\tau_{zx}(z) = -\int_{-\frac{h}{2}+o}^{z}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y}dz = \int_{z}^{+o+\frac{h}{2}}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y}dz \tag{62}$$

$$\tau_{yz}(z) = -\int_{-\frac{h}{2}+o}^{z}\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y}dz = \int_{z}^{+o+\frac{h}{2}}\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y}dz. \tag{63}$$

$$\underline{\underline{D}}_{123} = \left[\begin{array}{ccc} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{array}\right] \tag{64}$$

$$\left[\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array}\right] = \underline{\underline{T}}_1\left[\begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array}\right] \qquad \left[\begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{array}\right] = \underline{\underline{T}}_2\left[\begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array}\right] \tag{65}$$

$$\underline{\underline{\mathrm{T}}}_1 = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \tag{66}$$

$$\underline{\underline{\mathrm{T}}}_2 = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \tag{67}$$

$$m = \cos(\alpha) \qquad\qquad n = \sin(\alpha) \tag{68}$$

$$\underline{\underline{\mathrm{T}}}_1^{-1}(+\alpha) = \underline{\underline{\mathrm{T}}}_1(-\alpha) \qquad\qquad \underline{\underline{\mathrm{T}}}_2^{-1}(+\alpha) = \underline{\underline{\mathrm{T}}}_2(-\alpha) \tag{69}$$

$$\underline{\sigma} = \underline{\underline{\mathrm{D}}}\,\underline{\epsilon} \qquad\qquad \underline{\underline{\mathrm{D}}} \equiv \underline{\underline{\mathrm{D}}}_{xyz} = \underline{\underline{\mathrm{T}}}_1^{-1}\,\underline{\underline{\mathrm{D}}}_{123}\,\underline{\underline{\mathrm{T}}}_2 \tag{70}$$

$$\underline{\underline{\mathrm{G}}} = \begin{bmatrix} n^2 G_{z1} + m^2 G_{2z} & mnG_{z1} - mnG_{2z} \\ mnG_{z1} - mnG_{2z} & m^2 G_{z1} + n^2 G_{2z} \end{bmatrix}.$$

$$k_{\mathrm{x}}^* = \frac{12Fl}{Ebh^3} \tag{71}$$

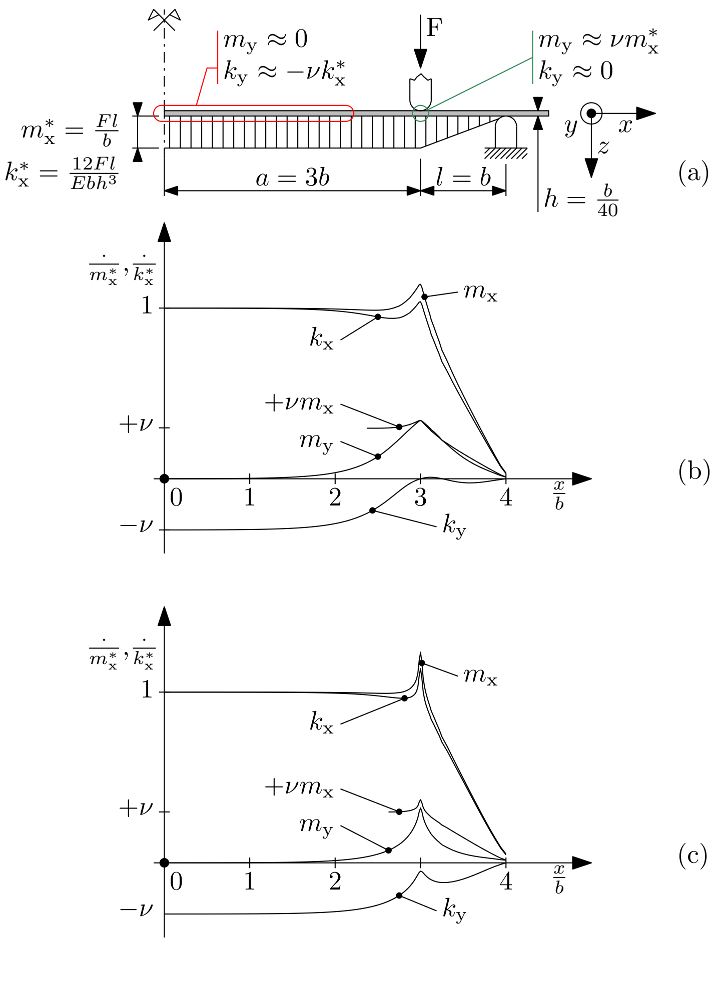
$$m_{\mathrm{x}} = m_{\mathrm{x}}^* \qquad m_{\mathrm{y}} = 0 \qquad \kappa_{\mathrm{x}} = k_{\mathrm{x}}^* \qquad \kappa_{\mathrm{y}} = -\nu k_{\mathrm{x}}^*,$$

$$m_{\mathrm{x}} = m_{\mathrm{x}}^* \qquad m_{\mathrm{y}} = \nu m_{\mathrm{x}}^* \qquad \kappa_{\mathrm{x}} = \left(1 - \nu^2\right) k_{\mathrm{x}}^* \qquad \kappa_{\mathrm{y}} = 0.$$

$$g(y) \geq 0 \tag{72}$$

$$f(y) \geq 0 \tag{73}$$

$$g(y) \cdot f(y) = 0, \tag{74}$$



$m_y \approx 0$
$k_y \approx -\nu k_x^*$
F
$m_y \approx \nu m_x^*$
$k_y \approx 0$
$m_x^* = \frac{Fl}{b}$
$k_x^* = \frac{12Fl}{Ebh^3}$
$a = 3b$
$l = b$
$h = \frac{b}{40}$
y
x
z
(a)
$\frac{\cdot}{m_x^*}, \frac{\cdot}{k_x^*}$
1
m_x
k_x
$+\nu m_x$
$+\nu$
m_y
0
1
2
3
4
$\frac{x}{b}$
$-\nu$
k_y
(b)
$\frac{\cdot}{m_x^*}, \frac{\cdot}{k_x^*}$
1
m_x
k_x
$+\nu m_x$
$+\nu$
m_y
0
1
2
3
4
$\frac{x}{b}$
$-\nu$
k_y
(c)

$$f(\xi,\eta) \overset{\text{def}}{=} \sum_i N_i(\xi,\eta) f_i \tag{75}$$

$$N_i(\xi,\eta) \overset{\text{def}}{=} \frac{1}{4}\left(1 \pm \xi\right)\left(1 \pm \eta\right), \tag{76}$$

$$\frac{\partial f}{\partial \xi} = \underbrace{\left(\frac{f_2 - f_1}{2}\right)}_{[\Delta f/\Delta\xi]_{12}} \underbrace{\left(\frac{1-\eta}{2}\right)}_{N_1+N_2} + \underbrace{\left(\frac{f_3 - f_4}{2}\right)}_{[\Delta f/\Delta\xi]_{43}} \underbrace{\left(\frac{1+\eta}{2}\right)}_{N_4+N_3} = a\eta + b \tag{77}$$

$$\frac{\partial f}{\partial \eta} = \left(\frac{f_4 - f_1}{2}\right)\left(\frac{1-\xi}{2}\right) + \left(\frac{f_3 - f_2}{2}\right)\left(\frac{1+\xi}{2}\right) = c\xi + d. \tag{78}$$

$$\begin{aligned} f(\xi,\eta) &= \begin{bmatrix} N_1(\xi,\eta) & \cdots & N_i(\xi,\eta) & \cdots & N_n(\xi,\eta) \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix} \\ &= \underline{\underline{\mathrm{N}}}\,(\xi,\eta)\,\underline{\mathrm{f}}\,, \end{aligned} \tag{79}$$

$$\underline{\mathrm{x}}\ \left(\underline{\xi}\right) = \underline{\mathrm{m}}\ \left(\underline{\xi}\right) = \sum_{i=1}^{4} N_i\left(\underline{\xi}\right)\ \underline{\mathrm{x}}_{\,i}, \tag{80}$$

$$\underline{\mathrm{m}}\ \left(\underline{\xi}\right) = \begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix}$$

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta) x_i \qquad y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta) y_i.$$

$$f(\xi,\eta) \overset{\text{def}}{=} \sum_i N_i(\xi,\eta) f_i \tag{81}$$

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underline{\underline{\mathrm{J}}}^{\top}(\xi,\eta;\underline{\mathrm{x}}_{\,i})} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \tag{82}$$

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \sum_i \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} f_i. \tag{83}$$

$$\underline{\underline{\mathrm{J}}}^\top(\xi, \eta) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{84}$$

$$= \sum_i \left(\begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 \\ \frac{\partial N_i}{\partial \eta} & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 & \frac{\partial N_i}{\partial \xi} \\ 0 & \frac{\partial N_i}{\partial \eta} \end{bmatrix} y_i \right) \tag{85}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \left(\underline{\underline{\mathrm{J}}}^\top\right)^{-1} \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial \xi} & \cdots \\ \cdots & \frac{\partial N_i}{\partial \eta} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ f_i \\ \vdots \end{bmatrix} \tag{86}$$

$$= \underbrace{\left(\underline{\underline{\mathrm{J}}}^\top\right)^{-1} \begin{bmatrix} \frac{\partial \underline{\mathrm{N}}}{\partial \xi} \\ \frac{\partial \underline{\mathrm{N}}}{\partial \eta} \end{bmatrix}}_{\underline{\underline{\mathrm{L}}}(\xi,\eta;\underline{\mathrm{x}}_i),\text{ or just } \underline{\underline{\mathrm{L}}}(\xi,\eta)} \underline{\mathrm{f}} \tag{87}$$

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i; \tag{88}$$

$$p(\xi) \stackrel{\text{def}}{=} a_m \xi^m + a_{m-1}\xi^{m-1} + \ldots + a_1 \xi + a_0$$

$$\int_{-1}^{1} p(\xi) d\xi = \sum_{j=0}^{m} \frac{(-1)^j + 1}{j+1} a_j$$

$$r\left(a_j, (\xi_i, w_i)\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} p(\xi_i) w_i - \int_{-1}^{1} p(\xi) d\xi \tag{89}$$

$$\left\{ \frac{\partial r\left(a_j, (\xi_i, w_i)\right)}{\partial a_j} = 0, \quad j = 0 \ldots m \right. \tag{90}$$

$$\int_a^b g(x) dx = \int_{-1}^{1} g\left(m(\xi)\right) \frac{dm}{d\xi} d\xi \approx \sum_{i=1}^{n} g\left(m(\xi_i)\right) \left.\frac{dm}{d\xi}\right|_{\xi=\xi_i} w_i. \tag{91}$$

$$m(x) = \underbrace{\left(\frac{1-\xi}{2}\right)}_{N_1} a + \underbrace{\left(\frac{1+\xi}{2}\right)}_{N_2} b.$$

$$\frac{dm}{d\xi} = \frac{dN_1}{d\xi}a + \frac{dN_2}{d\xi}b = \frac{b-a}{2}$$

$$\int_a^b g(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n} g\left(\frac{b+a}{2} + \frac{b-a}{2}\xi_i\right) w_i. \tag{92}$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \sum_{i=1}^{p}\sum_{j=1}^{q} f(\xi_i,\eta_j)\, w_i w_j \tag{93}$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \sum_{l=1}^{pq} f\left(\underline{\xi}_{\,l}\right) w_l \tag{94}$$

$$\underline{\xi}_{\,l} = (\xi_i,\eta_j)\,, \quad w_l = w_i w_j, \quad l = 1\ldots pq \tag{95}$$

$$\begin{aligned} dA_{xy} = \quad & \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P \quad\quad , \eta_P \quad\quad) & y(\xi_P \quad\quad , \eta_P \quad\quad) \\ 1 & x(\xi_P + d\xi, \eta_P \quad\quad) & y(\xi_P + d\xi, \eta_P \quad\quad) \\ 1 & x(\xi_P + d\xi, \eta_P + d\eta) & y(\xi_P + d\xi, \eta_P + d\eta) \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P + d\xi, \eta_P + d\eta) & y(\xi_P + d\xi, \eta_P + d\eta) \\ 1 & x(\xi_P \quad\quad , \eta_P + d\eta) & y(\xi_P \quad\quad , \eta_P + d\eta) \\ 1 & x(\xi_P \quad\quad , \eta_P \quad\quad) & y(\xi_P \quad\quad , \eta_P \quad\quad) \end{vmatrix}. \end{aligned} \tag{96}$$

$$\mathcal{A} = \frac{1}{2!}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad \mathcal{H} = \frac{1}{n!}\begin{vmatrix} 1 & \underline{\mathrm{x}}_{\,1} & \\ 1 & \underline{\mathrm{x}}_{\,2} & \\ \vdots & \vdots & \\ 1 & \underline{\mathrm{x}}_{\,n+1} & \end{vmatrix} \tag{97}$$

$$\begin{aligned} dA_{xy} \approx \quad & \frac{1}{2!}\begin{vmatrix} 1 & x & y \\ 1 & x + x_{,\xi}d\xi & y + y_{,\xi}d\xi \\ 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \\ 1 & x + x_{,\eta}d\eta & y + y_{,\eta}d\eta \\ 1 & x & y \end{vmatrix} \end{aligned}$$

$$dA_{xy} = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2}\begin{vmatrix} 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \\ 1 & x & y \end{vmatrix} d\xi d\eta$$

$$dA_{xy} = \frac{1}{2}\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2}\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta$$

$$dA_{xy} = \underbrace{\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix}}_{\left|J^{\mathrm{T}}(\xi_P,\eta_P;\underline{\mathrm{x}},\underline{\mathrm{y}})\right|} dA_{\xi\eta} \tag{98}$$

$$\iint_{A_{xy}} g(x,y)dA_{xy} = \int_{-1}^{1}\int_{-1}^{1} g\left(x\left(\xi,\eta\right), y\left(\xi,\eta\right)\right)\left|J(\xi,\eta)\right| d\xi d\eta, \tag{99}$$

$$\iint_{A_{xy}} g(\underline{\mathrm{x}})dA_{xy} \approx \sum_{l=1}^{pq} g\left(\underline{\mathrm{x}}\left(\underline{\xi}_l\right)\right)\left|J(\underline{\xi}_l)\right| w_l \tag{100}$$

$$dA_{xyz} = \sqrt{\begin{vmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{vmatrix}^2 + \begin{vmatrix} y_{,\xi} & y_{,\eta} \\ z_{,\xi} & z_{,\eta} \end{vmatrix}^2 + \begin{vmatrix} z_{,\xi} & z_{,\eta} \\ x_{,\xi} & x_{,\eta} \end{vmatrix}^2} d\xi d\eta \tag{101}$$

$$\underline{\mathrm{L}}\left(\xi,\eta;\underline{\mathrm{x}}_i\right) \approx ... \tag{102}$$

This is a four-node, thick-shell element with global displacements and rotations as degrees of freedom. Bilinear interpolation is used for the coordinates, displacements and the rotations. The membrane strains are obtained from the displacement field; the curvatures from the rotation field. The transverse shear strains are calculated at the middle of the edges and interpolated to the integration points. In this way, a very efficient and simple element is obtained which exhibits correct behavior in the limiting case of thin shells. The element can be used in curved shell analysis as well as in the analysis of complicated plate structures. For the latter case, the element is easy to use since connections between intersecting plates can be modeled without tying. Due to its simple formulation when compared to the standard higher order shell elements, it is less expensive and, therefore, very attractive in nonlinear analysis. The element is not very sensitive to distortion, particularly if the corner nodes lie in the same plane. All constitutive relations can be used with this element.

— MSC.Marc 2013.1 Documentation, vol. B, Element library.

$$\begin{bmatrix} X(\xi,\eta) \\ Y(\xi,\eta) \\ Z(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{n} N_i(\xi,\eta) \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} , \qquad \begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \\ z(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{n} N_i(\xi,\eta) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{103}$$

$$\begin{bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \\ w(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{104}$$

$$\begin{bmatrix} \theta(\xi,\eta) \\ \phi(\xi,\eta) \\ \psi(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} \theta_i \\ \phi_i \\ \psi_i \end{bmatrix} \tag{105}$$

$$\underline{u} = \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} \vdots \\ v_i \\ \vdots \end{bmatrix} \qquad \underline{w} = \begin{bmatrix} \vdots \\ w_i \\ \vdots \end{bmatrix}$$

$$\underline{\theta} = \begin{bmatrix} \vdots \\ \theta_i \\ \vdots \end{bmatrix} \qquad \underline{\phi} = \begin{bmatrix} \vdots \\ \phi_i \\ \vdots \end{bmatrix} \qquad \underline{\psi} = \begin{bmatrix} \vdots \\ \psi_i \\ \vdots \end{bmatrix}$$

$$u(\xi,\eta) = \underline{\underline{N}}(\xi,\eta)\,\underline{u} \qquad v(\xi,\eta) = \underline{\underline{N}}(\xi,\eta)\,\underline{v}$$

$$\underline{d}^\top = \begin{bmatrix} \underline{u}^\top & \underline{v}^\top & \underline{w}^\top & \underline{\theta}^\top & \underline{\phi}^\top & \underline{\psi}^\top \end{bmatrix} \tag{106}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \underbrace{\left(\underline{\underline{J}}'\right)^{-1} \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial \xi} & \cdots \\ \cdots & \frac{\partial N_i}{\partial \eta} & \cdots \end{bmatrix}}_{\underline{\underline{L}}(\xi,\eta;\underline{x}_i) \text{ or just } \underline{\underline{L}}(\xi,\eta)} \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix} \tag{107}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{L}}(\xi,\eta) & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{L}}(\xi,\eta) \end{bmatrix}}_{\underline{\underline{Q}}(\xi,\eta)} \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} \tag{108}$$

$$\begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underline{\underline{Q}}(\xi,\eta) \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{109}$$

$$\begin{bmatrix} e_x \\ e_y \\ g_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & +1 & 0 \end{bmatrix}}_{\underline{\underline{H}}'} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underline{\underline{H}}'\,\underline{\underline{Q}}(\xi,\eta) \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} \tag{110}$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & +1 \end{bmatrix}}_{\underline{\underline{H}}''} \begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underline{\underline{H}}''\,\underline{\underline{Q}}(\xi,\eta) \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{111}$$

$$\underline{\mathrm{e}} = \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{H}}}'\,\underline{\underline{\mathrm{Q}}}(\xi,\eta) & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{\mathrm{B}}}_e(\xi,\eta)} \underline{\mathrm{d}} \tag{112}$$

$$\underline{\kappa} = \underbrace{\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\mathrm{H}}}''\,\underline{\underline{\mathrm{Q}}}(\xi,\eta) & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)} \underline{\mathrm{d}}\,. \tag{113}$$

$$\underline{\epsilon}\,(\xi,\eta,z) = \left(\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + z\,\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\right)\,\underline{\mathrm{d}}\,; \tag{114}$$

$$\begin{bmatrix} \bar{\gamma}_{zx} \\ \bar{\gamma}_{yz} \end{bmatrix} = \underline{\underline{\mathrm{L}}}\,(\xi,\eta)\,\underline{\mathrm{w}} + \begin{bmatrix} \underline{\underline{0}} & +\underline{\underline{\mathrm{N}}}\,(\xi,\eta) \\ -\underline{\underline{\mathrm{N}}}\,(\xi,\eta) & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix}, \tag{115}$$

$$\begin{bmatrix} \bar{\gamma}_{zx} \\ \bar{\gamma}_{yz} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\mathrm{L}}}\,(\xi,\eta) & \begin{matrix} 0 \\ -\underline{\underline{\mathrm{N}}}\,(\xi,\eta) \end{matrix} & \begin{matrix} \underline{\underline{\mathrm{N}}}\,(\xi,\eta) \\ 0 \end{matrix} & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{\mathrm{B}}}_\gamma(\xi,\eta)} \underline{\mathrm{d}} \tag{116}$$

$$\underline{\mathrm{d}}^\top = \begin{bmatrix} \underline{\mathrm{u}}^\top & \underline{\mathrm{v}}^\top & \underline{\mathrm{w}}^\top & \underline{\theta}^\top & \underline{\phi}^\top & \underline{\psi}^\top \end{bmatrix} \tag{117}$$

$$\underline{\mathrm{G}}^\top = \begin{bmatrix} \underline{\mathrm{U}}^\top & \underline{\mathrm{V}}^\top & \underline{\mathrm{W}}^\top & \underline{\Theta}^\top & \underline{\Phi}^\top & \underline{\Psi}^\top \end{bmatrix} \tag{118}$$

$$\delta\,\Upsilon_{\mathrm{e}} = \delta\,\underline{\mathrm{d}}^\top\,\underline{\mathrm{G}}\,. \tag{119}$$

$$\underline{\sigma} = \underline{\underline{\mathrm{D}}}\,(z)\left(\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)z\right)\,\underline{\mathrm{d}} \tag{120}$$

$$\delta\,\underline{\epsilon} = \left(\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)z\right)\,\delta\,\underline{\mathrm{d}} \tag{121}$$

$$\underline{\mathrm{q}} = \left(\underline{\underline{\mathrm{a}}}\,\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{b}}}\,\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\right)\,\underline{\mathrm{d}} \tag{122}$$

$$\underline{\mathrm{m}} = \left(\underline{\underline{\mathrm{b}}}^\top\,\underline{\underline{\mathrm{B}}}_e(\xi,\eta) + \underline{\underline{\mathrm{c}}}\,\underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\right)\,\underline{\mathrm{d}}\,, \tag{123}$$

$$\delta\,\underline{\mathrm{e}} = \underline{\underline{\mathrm{B}}}_e(\xi,\eta)\,\delta\,\underline{\mathrm{d}} \tag{124}$$

$$\delta\,\underline{\kappa} = \underline{\underline{\mathrm{B}}}_\kappa(\xi,\eta)\,\delta\,\underline{\mathrm{d}}\,, \tag{125}$$

$$\begin{aligned}
\delta\,\Upsilon_{\mathrm{i}}^\dagger &= \iint_{\mathcal{A}} \int_h \delta\,\underline{\epsilon}^\top\,\underline{\sigma}\,dz d\mathcal{A} \\
&= \iint_{\mathcal{A}} \int_h \left(\left(\underline{\underline{\mathrm{B}}}_e + \underline{\underline{\mathrm{B}}}_\kappa z\right)\,\delta\,\underline{\mathrm{d}}\right)^\top\,\underline{\underline{\mathrm{D}}}\,\left(\underline{\underline{\mathrm{B}}}_e + \underline{\underline{\mathrm{B}}}_\kappa z\right)\,\underline{\mathrm{d}}\,dz d\mathcal{A} \\
&= \delta\,\underline{\mathrm{d}}^\top \left[\iint_{\mathcal{A}} \int_h \left(\underline{\underline{\mathrm{B}}}_e^\top + \underline{\underline{\mathrm{B}}}_\kappa^\top z\right)\,\underline{\underline{\mathrm{D}}}\,\left(\underline{\underline{\mathrm{B}}}_e + \underline{\underline{\mathrm{B}}}_\kappa z\right)\,dz d\mathcal{A}\right]\,\underline{\mathrm{d}} \\
&= \delta\,\underline{\mathrm{d}}^\top\,\underline{\underline{\mathrm{K}}}^\dagger\,\underline{\mathrm{d}}\,,
\end{aligned} \tag{126}$$

$$\begin{aligned}\delta\,\Upsilon_{\mathrm{i}}^{\dagger} &= \iint_{\mathcal{A}} \left(\delta\,\underline{\mathrm{e}}^{\top}\,\underline{\mathrm{q}} + \delta\,\underline{\kappa}^{\top}\,\underline{\mathrm{m}}\right) d\mathcal{A} \\ &= \delta\,\underline{\mathrm{d}}^{\top}\left[\iint_{\mathcal{A}} \begin{bmatrix}\underline{\underline{\mathrm{B}}}_{e}\\ \underline{\underline{\mathrm{B}}}_{\kappa}\end{bmatrix}^{\top}\begin{bmatrix}\underline{\underline{\mathrm{a}}} & \underline{\underline{\mathrm{b}}}\\ \underline{\underline{\mathrm{b}}}^{\mathrm{T}} & \underline{\underline{\mathrm{c}}}\end{bmatrix}\begin{bmatrix}\underline{\underline{\mathrm{B}}}_{e}\\ \underline{\underline{\mathrm{B}}}_{\kappa}\end{bmatrix} d\mathcal{A}\right]\underline{\mathrm{d}} \\ &= \delta\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}_{\sigma}\,\underline{\mathrm{d}}\,, \end{aligned} \tag{127}$$

$$\left\{\underline{\underline{\mathrm{a}}}, \underline{\underline{\mathrm{b}}}, \underline{\underline{\mathrm{c}}}\right\} = \int_{h} \underline{\underline{\mathrm{D}}}\,\left\{1, z, z^{2}\right\} dz,$$

$$\iiint_{\Omega} g(\xi,\eta,x,y,z)d\Omega = \tag{128}$$

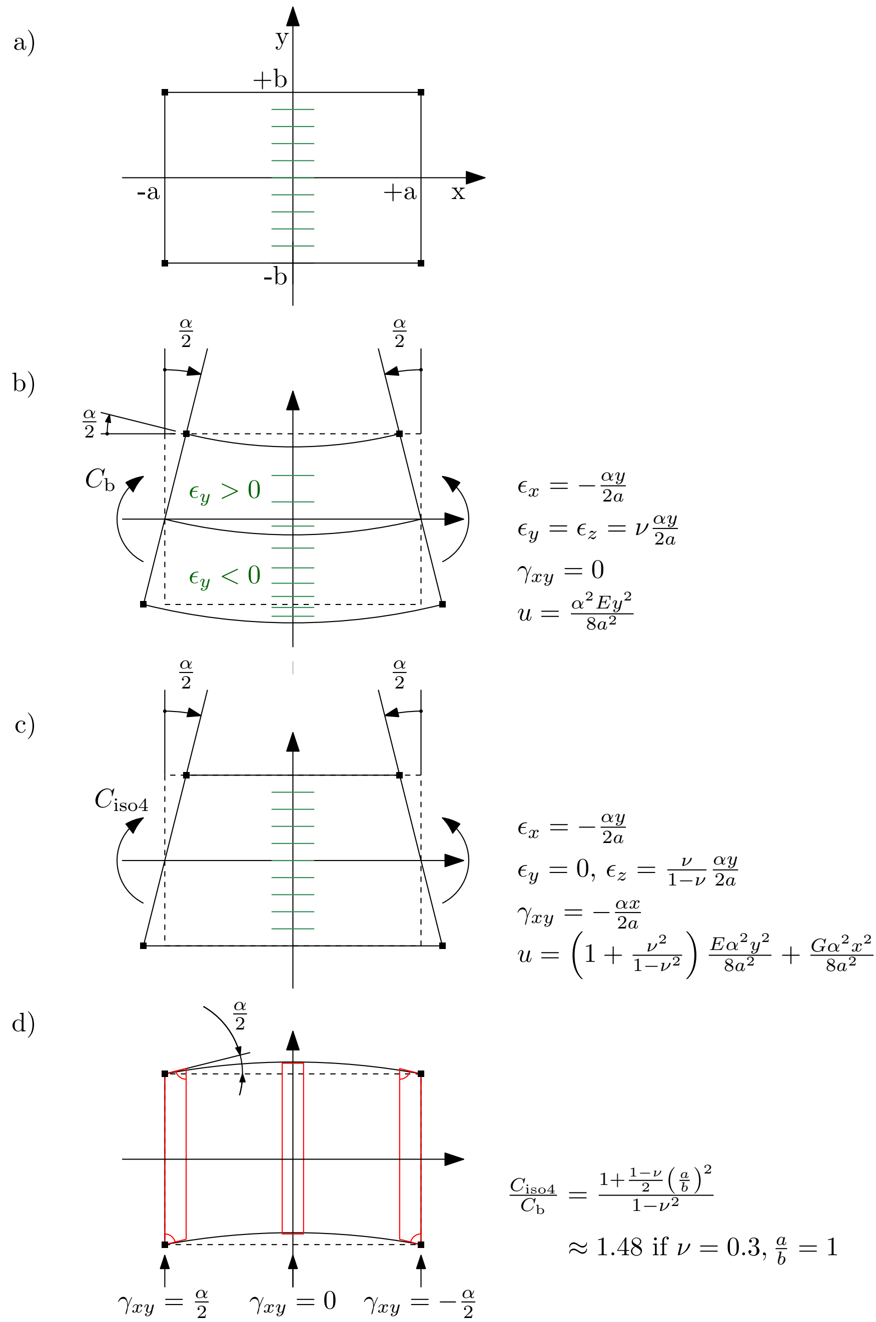
$$= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-\frac{h}{2}+o}^{+\frac{h}{2}+o} g(\xi,\eta,x(\xi,\eta),y(\xi,\eta),z)dz\,\left|\,\underline{\underline{\mathrm{J}}}\,(\xi,\eta)\right| d\xi d\eta,$$

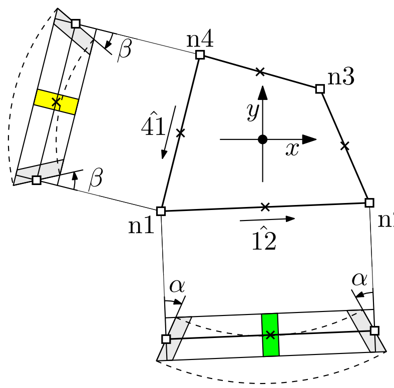
$$\begin{aligned}\delta\,\Upsilon_{\mathrm{i}}^{\ddagger} &= \iint_{\mathcal{A}} \delta\,\underline{\gamma}_{z}^{\top}\,\underline{\mathrm{q}}_{z} d\mathcal{A} \\ &= \delta\,\underline{\mathrm{d}}^{\top}\left[\iint_{\mathcal{A}} \underline{\underline{\mathrm{B}}}_{\gamma}^{\top}\,\underline{\underline{\Gamma}}\,\underline{\underline{\mathrm{B}}}_{\gamma} d\mathcal{A}\right]\underline{\mathrm{d}} \\ &= \delta\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}^{\ddagger}\,\underline{\mathrm{d}}\,. \end{aligned} \tag{129}$$

$$\begin{aligned}\delta\,\Upsilon_{\mathrm{i}} &= \delta\,\Upsilon_{\mathrm{i}}^{\dagger} + \delta\,\Upsilon_{\mathrm{i}}^{\ddagger} \\ &= \delta\,\underline{\mathrm{d}}^{\top}\left(\underline{\underline{\mathrm{K}}}^{\dagger} + \underline{\underline{\mathrm{K}}}^{\ddagger}\right)\underline{\mathrm{d}} \\ &= \delta\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}}\,. \end{aligned} \tag{130}$$

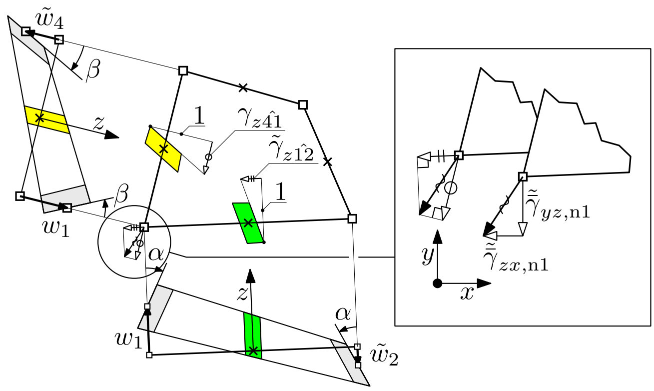
$$\delta\,\underline{\mathrm{d}}^{\top}\,\underline{\mathrm{G}} = \delta\,\Upsilon_{\mathrm{e}} = \delta\,\Upsilon_{\mathrm{i}} = \delta\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}}\,, \quad \forall\,\delta\,\underline{\mathrm{d}}\,, \tag{131}$$

$$\underline{\mathrm{G}} = \underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}}\,; \tag{132}$$


a)
y
x
+b
-b
-a
+a
b)
$\frac{\alpha}{2}$
C_b
$\epsilon_y > 0$
$\epsilon_y < 0$
$\epsilon_x = -\frac{\alpha y}{2a}$
$\epsilon_y = \epsilon_z = \nu \frac{\alpha y}{2a}$
$\gamma_{xy} = 0$
$u = \frac{\alpha^2 E y^2}{8a^2}$
c)
C_iso4
$\epsilon_x = -\frac{\alpha y}{2a}$
$\epsilon_y = 0,\ \epsilon_z = \frac{\nu}{1-\nu}\frac{\alpha y}{2a}$
$\gamma_{xy} = -\frac{\alpha x}{2a}$
$u = \left(1 + \frac{\nu^2}{1-\nu^2}\right)\frac{E\alpha^2 y^2}{8a^2} + \frac{G\alpha^2 x^2}{8a^2}$
d)
$\frac{\alpha}{2}$
$\frac{C_\mathrm{iso4}}{C_\mathrm{b}} = \frac{1+\frac{1-\nu}{2}\left(\frac{a}{b}\right)^2}{1-\nu^2}$
≈ 1.48 if $\nu = 0.3,\ \frac{a}{b} = 1$
$\gamma_{xy} = \frac{\alpha}{2}$
$\gamma_{xy} = 0$
$\gamma_{xy} = -\frac{\alpha}{2}$


n4
n3
n1
n2
$\hat{4}\hat{1}$
$\hat{1}\hat{2}$
x
y
α
β

(a)


$\tilde{w}_4$
w_1
$\tilde{w}_2$
z
β
α
$\gamma_{z\hat{4}\hat{1}}$
$\tilde{\gamma}_{z\hat{1}\hat{2}}$
1
$\tilde{\tilde{\gamma}}_{yz,\mathrm{n1}}$
$\tilde{\tilde{\gamma}}_{zx,\mathrm{n1}}$
x
y

(b)

$$\underline{\underline{\mathrm{S}}}\,(\xi,\eta,z) = \begin{bmatrix} \dots & \hat{u}_i(\xi,\eta,z) & \dots \\ \dots & \hat{v}_i(\xi,\eta,z) & \dots \\ \dots & \hat{w}_i(\xi,\eta,z) & \dots \end{bmatrix} \tag{133}$$

$$\underline{\mathrm{u}}\,(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}\,(\xi,\eta,z)\,\underline{\mathrm{d}}\,. \tag{134}$$

$$\dot{\underline{\mathrm{u}}}\,(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}\,(\xi,\eta,z)\,\dot{\underline{\mathrm{d}}} \tag{135}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \dot{\underline{\mathrm{u}}}^\top\,\dot{\underline{\mathrm{u}}}\,\rho d\Omega \tag{136}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \left[\underline{\underline{\mathrm{S}}}\,\dot{\underline{\mathrm{d}}}\right]^\top \left[\underline{\underline{\mathrm{S}}}\,\dot{\underline{\mathrm{d}}}\right]\rho d\Omega, \tag{137}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\,\dot{\underline{\mathrm{d}}}^\top \left[\iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\underline{\mathrm{S}}}\,\rho d\Omega\right]\dot{\underline{\mathrm{d}}} = \frac{1}{2}\,\dot{\underline{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\dot{\underline{\mathrm{d}}}\,. \tag{138}$$

$$\underline{\underline{\mathrm{M}}} = \iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\underline{\mathrm{S}}}\,\rho d\Omega, \tag{139}$$

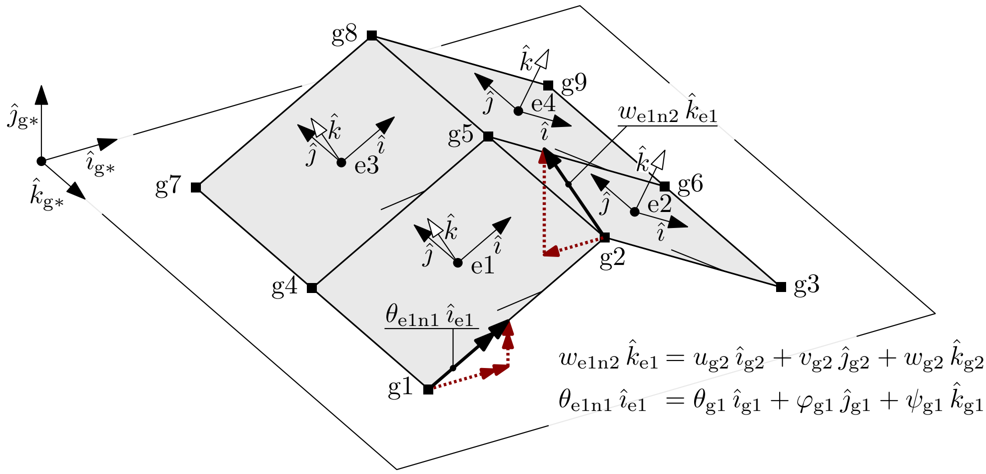
$$\begin{aligned} \dot{\underline{\mathrm{d}}}^\top\,\underline{\mathrm{G}} = \frac{dE_{\mathrm{kin}}}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\,\dot{\underline{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\dot{\underline{\mathrm{d}}}\right) \\ &= \frac{1}{2}\left(\ddot{\underline{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\dot{\underline{\mathrm{d}}} + \dot{\underline{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\ddot{\underline{\mathrm{d}}}\right) \\ &= \dot{\underline{\mathrm{d}}}^\top\,\underline{\underline{\mathrm{M}}}\,\ddot{\underline{\mathrm{d}}}\,. \end{aligned}$$

$$\underline{\mathrm{G}} = \underline{\underline{\mathrm{M}}}\,\ddot{\underline{\mathrm{d}}} \tag{140}$$

$$\delta\,\underline{\mathrm{u}}\,(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}\,(\xi,\eta,z)\delta\,\underline{\mathrm{d}}\,, \tag{141}$$

$$\begin{aligned} \delta\,\underline{\mathrm{d}}^\top\,\underline{\mathrm{F}} &= \iiint_\Omega (\delta\,\underline{\mathrm{u}})^\top\,\underline{\mathrm{p}}\,d\Omega \\ &= \iiint_\Omega \left(\underline{\underline{\mathrm{S}}}\,\delta\,\underline{\mathrm{d}}\right)^\top\,\underline{\mathrm{p}}\,d\Omega \\ &= \delta\,\underline{\mathrm{d}}^\top \iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\mathrm{p}}\,d\Omega, \end{aligned}$$

$$\underline{\mathrm{F}} = \iiint_\Omega \underline{\underline{\mathrm{S}}}^\top\,\underline{\mathrm{p}}\,d\Omega \tag{142}$$



$$\underline{\mathrm{G}}_{\,\mathrm{e}j} = \underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{e}j} \tag{143}$$

$$\underline{\mathrm{d}}_{\,\mathrm{g}l} = \begin{bmatrix} u_{\mathrm{g}l} \\ v_{\mathrm{g}l} \\ w_{\mathrm{g}l} \\ \theta_{\mathrm{g}l} \\ \varphi_{\mathrm{g}l} \\ \psi_{\mathrm{g}l} \end{bmatrix}. \tag{144}$$

$$\underline{\mathrm{d}}_{\,\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{d}}_{\,\mathrm{g1}}^{\top} & \underline{\mathrm{d}}_{\,\mathrm{g2}}^{\top} & \cdots & \underline{\mathrm{d}}_{\,\mathrm{g}l}^{\top} & \cdots & \underline{\mathrm{d}}_{\,\mathrm{g}n}^{\top} \end{bmatrix} \tag{145}$$

$$\underline{\mathrm{F}}_{\,\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{F}}_{\,\mathrm{g1}}^{\top} & \underline{\mathrm{F}}_{\,\mathrm{g2}}^{\top} & \cdots & \underline{\mathrm{F}}_{\,\mathrm{g}l}^{\top} & \cdots & \underline{\mathrm{F}}_{\,\mathrm{g}n}^{\top} \end{bmatrix}; \tag{146}$$

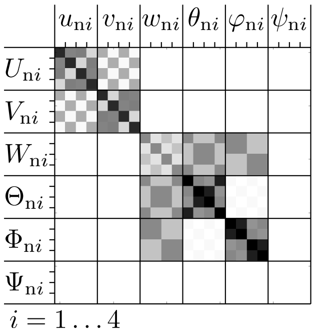
$$\underline{\mathrm{R}}_{\,\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{R}}_{\,\mathrm{g1}}^{\top} & \underline{\mathrm{R}}_{\,\mathrm{g2}}^{\top} & \cdots & \underline{\mathrm{R}}_{\,\mathrm{g}l}^{\top} & \cdots & \underline{\mathrm{R}}_{\,\mathrm{g}n}^{\top} \end{bmatrix} \tag{147}$$

$$w_{\mathrm{e1n2}} = \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle u_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle v_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle w_{\mathrm{g2}} \tag{148}$$

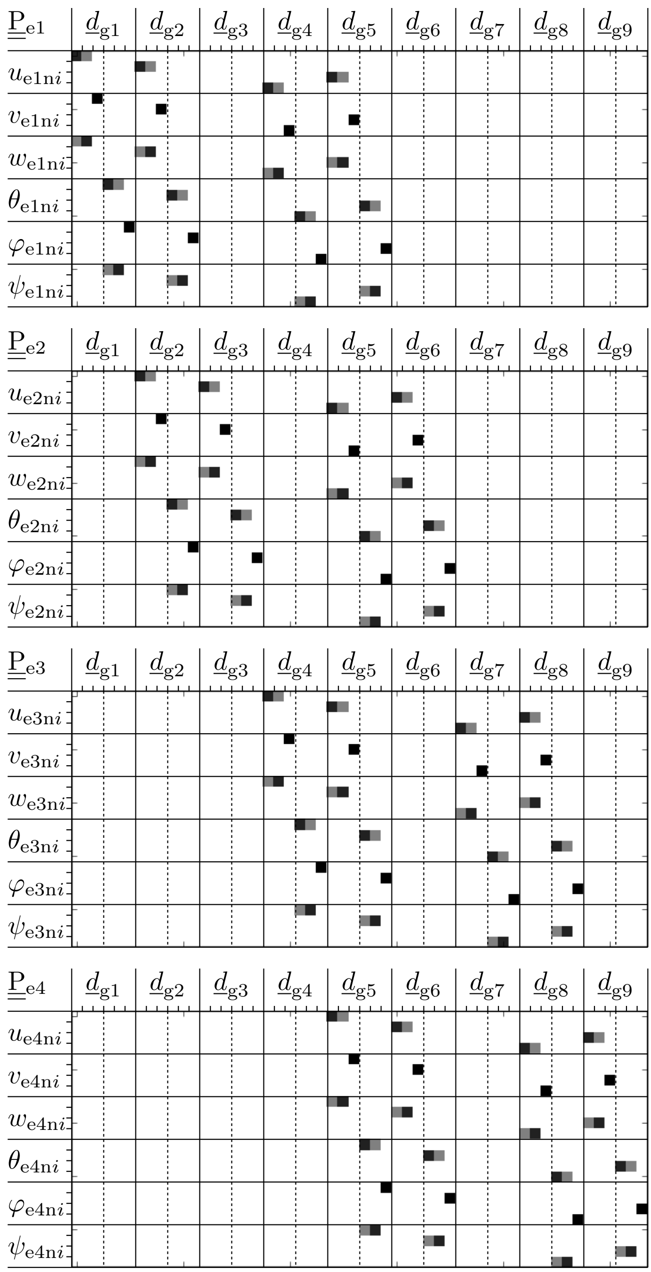
$$\theta_{\mathrm{e1n1}} = \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \theta_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \phi_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle \psi_{\mathrm{g1}} \tag{149}$$

$$\begin{aligned}
\left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{10,7} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{13,4} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{10,8} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{13,5} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{10,9} &= \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\,\mathrm{e1}}\right]_{13,6} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle,
\end{aligned}$$

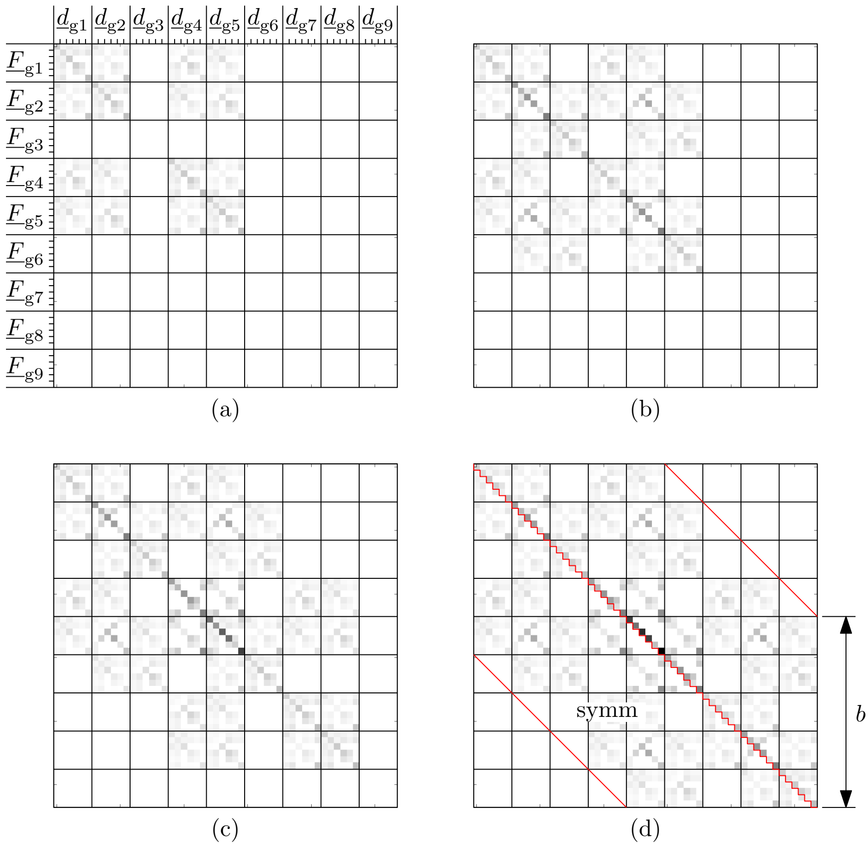
node	X	Y	Z
g1	$-ac$	0	$+a$
g2	0	$+as$	$+a$
g3	$+ac$	0	$+a$
g4	$-ac$	0	0
g5	0	$+as$	0
g6	$+ac$	0	0
g7	$-ac$	0	$-a$
g8	0	$+as$	$-a$
g9	$+ac$	0	$-a$



	n1	n2	n3	n4
e1	g1	g2	g5	g4
e2	g2	g3	g6	g5
e3	g4	g5	g8	g7
e4	g5	g6	g9	g8



Pe1
dg1
dg2
dg3
dg4
dg5
dg6
dg7
dg8
dg9
ue1ni
ve1ni
we1ni
θe1ni
φe1ni
ψe1ni
Pe2
ue2ni
ve2ni
we2ni
θe2ni
φe2ni
ψe2ni
Pe3
ue3ni
ve3ni
we3ni
θe3ni
φe3ni
ψe3ni
Pe4
ue4ni
ve4ni
we4ni
θe4ni
φe4ni
ψe4ni



$$\underline{\mathrm{d}}_{\,\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \quad \forall j. \tag{150}$$

$$\underline{\mathrm{G}}_{\,\mathrm{e}j} = \underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \quad \forall j; \tag{151}$$

$$\delta\,\underline{\mathrm{d}}_{\,\mathrm{g}}^{\top}\,\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \left(\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\delta\,\underline{\mathrm{d}}_{\,\mathrm{g}}\right)^{\top}\,\underline{\mathrm{G}}_{\,\mathrm{e}j}, \quad \forall\ \delta\,\underline{\mathrm{d}}_{\,\mathrm{g}} \tag{152}$$

$$\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\mathrm{G}}_{\,\mathrm{e}j} \tag{153}$$

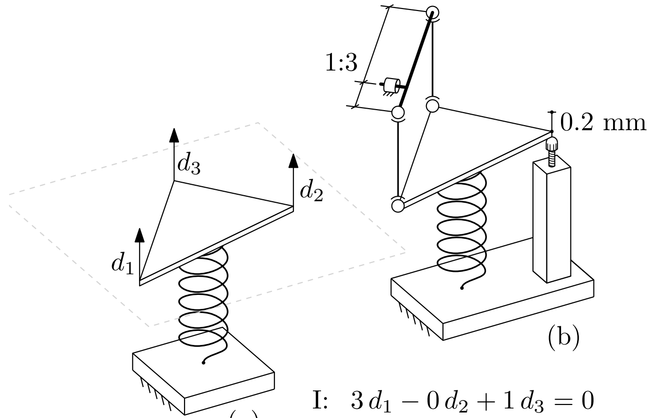
$$\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}; \tag{154}$$

$$\underline{\mathrm{G}}_{\,\mathrm{g}} = \sum_j \underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \left(\sum_j \underbrace{\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}}_{\underline{\underline{\mathrm{K}}}_{\,\mathrm{g}\leftarrow\mathrm{e}j}}\right)\underline{\mathrm{d}}_{\,\mathrm{g}} = \underline{\underline{\mathrm{K}}}_{\,\mathrm{g}}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \tag{155}$$

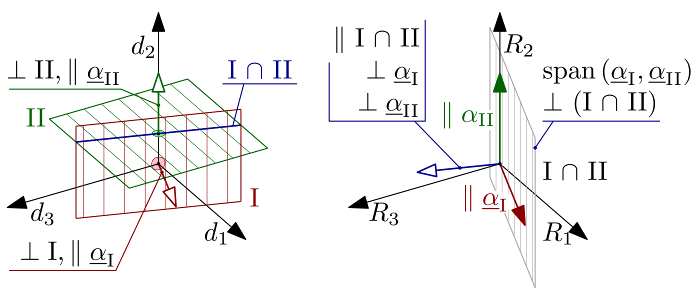
$$b_{\mathrm{e}j} = (i_{\max} - i_{\min} + 1)\, l, \tag{156}$$

$$b = \max_{\mathrm{e}j} b_{\mathrm{e}j} \tag{157}$$

$$\underline{\mathrm{F}}_{\,\mathrm{g}} = \sum_j \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top} \underline{\mathrm{F}}_{\,\mathrm{e}j}; \tag{158}$$



$$\begin{aligned} \text{I:} &\quad 3\,d_1 - 0\,d_2 + 1\,d_3 = 0 \\ \text{II:} &\quad 0\,d_1 + 1\,d_2 + 0\,d_3 = 0.2 \end{aligned}$$



(a) configuration space (b) reaction space

$$\sum_i \alpha_{ji} d_i = \underline{\alpha}_j^\top \underline{\mathrm{d}} = \beta_j$$

$$\underline{\alpha}_{\mathrm{I}}^\top = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \qquad \beta_{\mathrm{I}} = 0$$

$$\underline{\alpha}_{\mathrm{II}}^\top = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad \beta_{\mathrm{II}} = 0.2$$

$$\sum_i \alpha_{ji} d_i = \underline{\alpha}_j^\top \underline{\mathrm{d}} = \beta_j, \quad j = 1 \ldots m \tag{159}$$

$$\underline{\underline{\mathcal{L}}}^\top \underline{\mathrm{d}} = \underline{\beta}\,. \tag{160}$$

$$\underline{\underline{\mathcal{L}}}^\top \delta\,\underline{\mathrm{d}} = \underline{0}\,, \tag{161}$$

$$\underline{\mathrm{R}} = -\underline{\underline{\mathcal{L}}}\,\underline{\ell}\,, \tag{162}$$

$$\underline{\mathrm{R}}^{\,j} = -\begin{bmatrix} \vdots \\ \alpha_{ji} \\ \vdots \end{bmatrix} \ell_j \tag{163}$$

$$\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} \;=\; \underline{\mathrm{F}} \,+\, \underline{\mathrm{R}}\,. \tag{164}$$

$$\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} \,+\, \underline{\underline{\mathcal{L}}}\,\underline{\ell} \;=\; \underline{\mathrm{F}}$$

$$\begin{bmatrix} \underline{\underline{\mathrm{K}}} & \underline{\underline{\mathcal{L}}} \\ \underline{\underline{\mathcal{L}}}^{\top} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{d}} \\ \underline{\ell} \end{bmatrix} = \begin{bmatrix} \underline{\mathrm{F}} \\ \underline{\beta} \end{bmatrix}, \tag{165}$$

$$\frac{1}{2}\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} \,-\, \underline{\mathrm{d}}^{\top}\,\underline{\mathrm{F}} \,+\, \underline{\ell}^{\top}\left(\underline{\underline{\mathcal{L}}}^{\top}\,\underline{\mathrm{d}} \,-\, \underline{\beta}\right), \tag{166}$$

$$\frac{1}{2}\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} \,-\, \underline{\mathrm{d}}^{\top}\,\underline{\mathrm{F}}$$

$$\underline{\underline{\mathcal{L}}}^{\top}\,\underline{\mathrm{d}} \,-\, \underline{\beta} \;=\; \underline{0}$$

$$\underline{\mathrm{R}} \;=\, -\,\underline{\underline{\mathcal{L}}}\;\underline{\ell}^{\,*}.$$

$$d_k = \sum_{i \neq k} \left(-\frac{\alpha_{ji}}{\alpha_{jk}}\right) d_i + \left(\frac{\beta_j}{\alpha_{jk}}\right), \quad j = 1 \ldots m \tag{167}$$

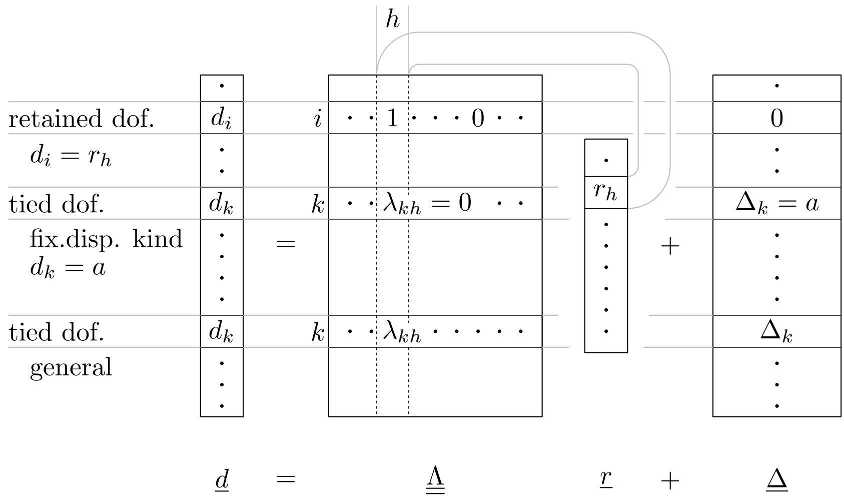
$$\begin{bmatrix} \underline{\mathrm{r}} \\ \underline{\mathrm{t}} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{E}}}_{\,\mathrm{R}} \\ \underline{\underline{\mathrm{E}}}_{\,\mathrm{T}} \end{bmatrix}}_{\underline{\underline{\mathrm{E}}}} \underline{\mathrm{d}} \qquad\qquad \underline{\mathrm{d}} \;=\; \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{E}}}_{\,\mathrm{R}}^{\top} & \underline{\underline{\mathrm{E}}}_{\,\mathrm{T}}^{\top} \end{bmatrix}}_{\underline{\underline{\mathrm{E}}}^{\top} \equiv \underline{\underline{\mathrm{E}}}^{-1}} \begin{bmatrix} \underline{\mathrm{r}} \\ \underline{\mathrm{t}} \end{bmatrix},$$

$$\underline{\mathrm{r}} \;=\; \underline{\underline{\mathrm{E}}}_{\,\mathrm{R}}\,\underline{\mathrm{d}} \qquad\qquad \underline{\mathrm{t}} \;=\; \underline{\underline{\mathrm{E}}}_{\,\mathrm{T}}\,\underline{\mathrm{d}}\,,$$

$$t_j = \sum_{h=1}^{n-m} \lambda_{jh} r_h + \delta_j, \quad j = 1 \ldots m \tag{168}$$

$$\underline{\mathrm{t}} \;=\; \underline{\underline{\lambda}}\;\underline{\mathrm{r}} \,+\, \underline{\delta}\,, \tag{169}$$

$$\begin{aligned} \underline{\mathrm{d}} \;&=\; \left(\underline{\underline{\mathrm{E}}}^{\top}\begin{bmatrix} \underline{\underline{\mathrm{I}}} \\ \underline{\underline{\lambda}} \end{bmatrix}\right)\underline{\mathrm{r}} \,+\, \left(\underline{\underline{\mathrm{E}}}^{\top}\begin{bmatrix} \underline{0} \\ \underline{\delta} \end{bmatrix}\right) \\ &=\; \underline{\underline{\Lambda}}\;\underline{\mathrm{r}} \,+\, \underline{\Delta}\,; \end{aligned} \tag{170}$$



$$\delta \, \underline{\mathrm{d}} = \underline{\underline{\Lambda}} \, \delta \, \underline{\mathrm{r}} = \underline{\Lambda}_{1} \, \delta r_1 + \underline{\Lambda}_{2} \, \delta r_2 + \ldots + \underline{\Lambda}_{n-m} \, \delta r_{n-m} \tag{171}$$

$$\langle \underline{\Lambda}_{h}, \underline{\mathrm{R}} \rangle = 0 \quad h = 1 \ldots n-m, \tag{172}$$

or, equivalently,

$$\underline{\underline{\Lambda}}^{\top} \underline{\mathrm{R}} = \underline{0}\,. \tag{173}$$

$$\underline{\underline{\mathrm{K}}} \left(\underline{\underline{\Lambda}} \, \underline{\mathrm{r}} + \underline{\Delta} \right) = \underline{\mathrm{F}} + \underline{\mathrm{R}} \tag{174}$$

$$\underline{\underline{\mathrm{K}}} \, \underline{\underline{\Lambda}} \, \underline{\mathrm{r}} = \left(\underline{\mathrm{F}} - \underline{\underline{\mathrm{K}}} \, \underline{\Delta} \right) + \underline{\mathrm{R}}\,, \tag{175}$$

$$\underbrace{\underline{\underline{\Lambda}}^{\top} \underline{\underline{\mathrm{K}}} \, \underline{\underline{\Lambda}}}_{\underline{\underline{\mathrm{K}}}_{\mathrm{R}}} \, \underline{\mathrm{r}} = \underbrace{\underline{\underline{\Lambda}}^{\top} \left(\underline{\mathrm{F}} - \underline{\underline{\mathrm{K}}} \, \underline{\Delta} \right)}_{\underline{\mathrm{F}}_{\mathrm{R}}} + \underbrace{\underline{\underline{\Lambda}}^{\top} \underline{\mathrm{R}}}_{=0}\,, \tag{176}$$

$$\underline{\underline{\mathrm{K}}}_{\mathrm{R}} \, \underline{\mathrm{r}} = \underline{\mathrm{F}}_{\mathrm{R}} \tag{177}$$

$$\underline{\mathrm{d}}^{*} = \underline{\underline{\Lambda}} \, \underline{\mathrm{r}}^{*} + \underline{\Delta}\,; \tag{178}$$

$$\underline{\mathrm{R}}^{*} = \underline{\underline{\mathrm{K}}} \left(\underline{\underline{\Lambda}} \, \underline{\mathrm{r}}^{*} + \underline{\Delta} \right) - \underline{\mathrm{F}}\,. \tag{179}$$

$$\underline{\mathrm{d}}^{*}_{\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\mathrm{e}j} \, \underline{\mathrm{d}}^{*}. \tag{180}$$

$$\underline{\mathrm{e}} = \underline{\underline{\mathrm{B}}}^{e}_{\mathrm{e}j}(\xi, \eta) \, \underline{\mathrm{d}}^{*}_{\mathrm{e}j} \qquad \underline{\kappa} = \underline{\underline{\mathrm{B}}}^{\kappa}_{\mathrm{e}j}(\xi, \eta) \, \underline{\mathrm{d}}^{*}_{\mathrm{e}j} \tag{181}$$

$$\underline{\epsilon} = \left(\underline{\underline{\mathrm{B}}}^{e}_{\mathrm{e}j}(\xi, \eta) + \underline{\underline{\mathrm{B}}}^{\kappa}_{\mathrm{e}j}(\xi, \eta) z \right) \, \underline{\mathrm{d}}^{*}_{\mathrm{e}j}. \tag{182}$$

$$\underline{\gamma}_{z} = \underline{\underline{\mathrm{B}}}^{\gamma}_{\mathrm{e}j}(\xi, \eta) \, \underline{\mathrm{d}}^{*}_{\mathrm{e}j}. \tag{183}$$

$$\begin{bmatrix} u_i \\ v_i \\ w_i \\ \theta_i \\ \phi_i \\ \psi_i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & +(z_i - z_C) & -(y_i - y_C) \\ 0 & 1 & 0 & -(z_i - z_C) & 0 & +(x_i - x_C) \\ 0 & 0 & 1 & +(y_i - y_C) & -(x_i - x_C) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{\mathrm{L}}}_i} \cdot \begin{bmatrix} u_C \\ v_C \\ w_C \\ \theta_C \\ \phi_C \\ \psi_C \end{bmatrix} \tag{184}$$

$$\begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} = q_i \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 0 & d & -f \\ -d & 0 & e \\ f & -e & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \right), \tag{185}$$

$$\begin{bmatrix} U_C \\ V_C \\ W_C \end{bmatrix} = \sum_i \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix}, \qquad \begin{bmatrix} \Theta_C \\ \Phi_C \\ \Psi_C \end{bmatrix} = \sum_i \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} \wedge \begin{bmatrix} x_i - x_C \\ y_i - y_C \\ z_i - z_C \end{bmatrix} \tag{186}$$

$$\underline{\ddot{\mathrm{d}}} = \underbrace{\begin{bmatrix} \cdots & \underline{\mathrm{t}}_l & \cdots \end{bmatrix}}_{\underline{\underline{\mathrm{T}}}} \underbrace{\begin{bmatrix} \vdots \\ \alpha_l \\ \vdots \end{bmatrix}}_{\underline{\alpha}}, \tag{187}$$

$$\underline{\underline{\mathrm{T}}}^{\top} \underline{\underline{\mathrm{M}}} \, \underline{\underline{\mathrm{T}}} \, \underline{\alpha} = \underline{\underline{\mathrm{T}}}^{\top} \underline{\mathrm{F}} \tag{188}$$

$$\underline{\underline{\mathrm{M}}} \, \underline{\underline{\mathrm{T}}} \, \underline{\alpha} = \underline{\mathrm{F}} \; [+ \underline{\mathrm{R}}_l]$$

$$\underline{\underline{\mathrm{K}}} \, \underline{\mathrm{d}} = \underline{\mathrm{F}} - \underline{\underline{\mathrm{M}}} \, \underline{\underline{\mathrm{T}}} \, \underline{\alpha}, \tag{189}$$

$$\underline{\underline{\mathrm{M}}}\,\ddot{\underline{\mathrm{d}}} + \underline{\underline{\mathrm{C}}}\,\dot{\underline{\mathrm{d}}} + \underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} = \underline{\mathrm{f}}\,(t), \quad \underline{\mathrm{d}} = \underline{\mathrm{d}}\,(t) \tag{190}$$

$$\underline{\mathrm{f}}\,(t) = \frac{\bar{\underline{\mathrm{f}}}\,e^{j\omega t} + \bar{\underline{\mathrm{f}}}\,^{*}e^{-j\omega t}}{2} = \mathrm{Re}(\bar{\underline{\mathrm{f}}}\,e^{j\omega t}) \tag{191}$$

$$\underline{\mathrm{f}}\,(t) = \bar{\underline{\mathrm{f}}}\,e^{j\omega t} \tag{192}$$

$$\mathrm{Re}(\bar{\underline{\mathrm{f}}}\,e^{j\omega t}) = \mathrm{Re}(\bar{\underline{\mathrm{f}}}\,)\cos\omega t - \mathrm{Im}(\bar{\underline{\mathrm{f}}}\,)\sin\omega t \tag{193}$$

$$\underline{\mathrm{d}}\,(t) = \bar{\underline{\mathrm{d}}}\,e^{j\omega t} \tag{194}$$

$$\left(-\omega^2\,\underline{\underline{\mathrm{M}}} + j\omega\,\underline{\underline{\mathrm{C}}} + \underline{\underline{\mathrm{K}}}\right)\,\bar{\underline{\mathrm{d}}} = \bar{\underline{\mathrm{f}}} \tag{195}$$

$$\left(-\omega^2\,\underline{\underline{\mathrm{M}}} + \underline{\underline{\mathrm{K}}}\right)\,\bar{\underline{\mathrm{d}}} = \underline{0} \tag{196}$$

$$\left(\underline{\underline{\mathrm{M}}}\,^{-1}\,\underline{\underline{\mathrm{K}}} - \omega^2\,\underline{\underline{\mathrm{I}}}\right)\,\hat{\underline{\mathrm{d}}} = \underline{0}\,; \tag{197}$$

$$m_i = \hat{\underline{\mathrm{d}}}\,_i^\top\,\underline{\underline{\mathrm{M}}}\,\hat{\underline{\mathrm{d}}}\,_i = 1 \tag{198}$$

$$\underline{\mathrm{x}}\,(t) = a\,\hat{\underline{\mathrm{d}}}\,_i\sin(\omega_i t) \tag{199}$$

$$f(t) = \hat{\underline{\mathrm{f}}}\,\cos(\omega_i t), \tag{200}$$

$$\underbrace{\left(-\omega_i^2\,\underline{\underline{\mathrm{M}}} + \underline{\underline{\mathrm{K}}}\right)\,\hat{\underline{\mathrm{d}}}\,_i}_{=\underline{0}}\,a_i\sin(\omega_i t) + \omega_i a_i\,\underline{\underline{\mathrm{C}}}\,\hat{\underline{\mathrm{d}}}\,_i\cos(\omega_i t) = \bar{\underline{\mathrm{f}}}\,\cos(\omega_i t). \tag{201}$$

$$a_i = \frac{\hat{\underline{\mathrm{d}}}\,^\top\,\bar{\underline{\mathrm{f}}}}{\omega_i\,\hat{\underline{\mathrm{d}}}\,^\top\,\underline{\underline{\mathrm{C}}}\,\hat{\underline{\mathrm{d}}}\,_i} \tag{202}$$

$$\hat{\underline{\mathrm{d}}}\,_j^\top\,\underline{\underline{\mathrm{M}}}\,\hat{\underline{\mathrm{d}}}\,_i = m_i\delta_{ij} \qquad \hat{\underline{\mathrm{d}}}\,_j^\top\,\underline{\underline{\mathrm{K}}}\,\hat{\underline{\mathrm{d}}}\,_i = m_i\omega_i^2\delta_{ij} \tag{203}$$

$$\underline{\underline{\Xi}} = \left[\hat{\underline{\mathrm{d}}}\,_1 \quad \cdots \quad \hat{\underline{\mathrm{d}}}\,_l \quad \cdots \quad \hat{\underline{\mathrm{d}}}\,_m\right], \tag{204}$$

$$\bar{\underline{\mathrm{d}}} = \underline{\underline{\Xi}}\,\bar{\underline{\xi}} \tag{205}$$

.

$$\underline{\underline{\Xi}}\,^\top\,\underline{\underline{\mathrm{M}}}\,\underline{\underline{\Xi}} = \underline{\underline{\mathrm{I}}} \qquad \underline{\underline{\Xi}}\,^\top\,\underline{\underline{\mathrm{K}}}\,\underline{\underline{\Xi}} = \underline{\underline{\Omega}} = \mathrm{diag}(\omega_l^2); \tag{206}$$

$$\underline{\underline{\mathrm{C}}} = \alpha\,\underline{\underline{\mathrm{M}}} + \beta\,\underline{\underline{\mathrm{K}}} \tag{207}$$

$$\underline{\underline{\Xi}}\,^\top\left(-\omega^2\,\underline{\underline{\mathrm{M}}} + j\omega\,\underline{\underline{\mathrm{C}}} + \underline{\underline{\mathrm{K}}}\right)\,\underline{\underline{\Xi}}\,\bar{\underline{\xi}} = \underline{\underline{\Xi}}\,^\top\,\bar{\underline{\mathrm{f}}} \tag{208}$$

$$\left(-\omega^2\,\underline{\underline{\mathrm{I}}} + j\omega\left(\alpha\,\underline{\underline{\mathrm{I}}} + \beta\,\underline{\underline{\Omega}}\right) + \underline{\underline{\Omega}}\right)\,\bar{\underline{\xi}} = \underline{\underline{\Xi}}\,^\top\,\bar{\underline{\mathrm{f}}}\,, \tag{209}$$

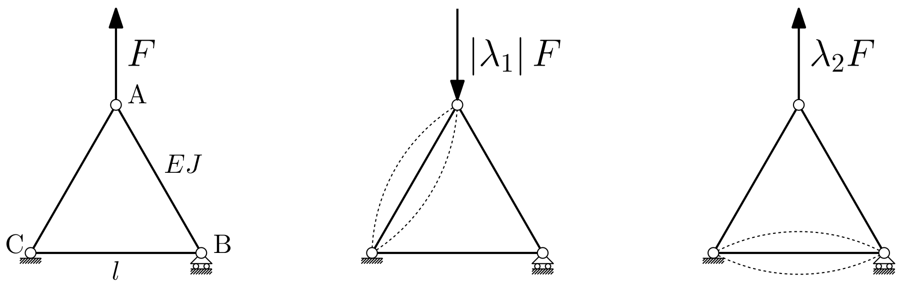
$$\left(-\omega^2 + j\omega\left(\alpha + \beta\omega_l^2\right) + \omega_l^2\right)\xi_l = q_l, \quad j = 1 \ldots m \tag{210}$$

$$\begin{aligned}
\xi_l(t) &= \mathrm{Re}(\bar{\xi}_l)\cos\omega t - \mathrm{Im}(\bar{\xi}_l)\sin\omega t \\
&= \left|\bar{\xi}_l\right|\cos\left(\omega t + \psi_l - \phi_l\right)
\end{aligned}$$

$$a_l = 1 - r_l^2 \qquad\qquad b_l = 2\zeta_l r_l \qquad\qquad r_l = \frac{\omega}{\omega_l}$$

$$\begin{aligned}
\left|\bar{\xi}_l\right| &= \frac{|\bar{q}_l|}{\omega_l^2}\frac{1}{\sqrt{a_l^2 + b_l^2}} \\
\psi_l &= \arg(\bar{q}_l) \\
\phi_l &= \arg(a_l + jb_l)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}(\bar{\xi}_l) &= \frac{1}{\omega_l^2}\frac{a_l\,\mathrm{Re}(\bar{q}_l) + b_l\,\mathrm{Im}(\bar{q}_l)}{a_l^2 + b_l^2} \\
\mathrm{Im}(\bar{\xi}_l) &= \frac{1}{\omega_l^2}\frac{a_l\,\mathrm{Im}(\bar{q}_l) - b_l\,\mathrm{Re}(\bar{q}_l)}{a_l^2 + b_l^2}.
\end{aligned}$$



$N_{AB} = N_{CA} = +\frac{F}{\sqrt{3}}$

$N_{BC} = -\frac{F}{2\sqrt{3}}$

$N_{\mathrm{crit},i} = -\frac{\pi^2 i^2 EJ}{l^2}$

$\lambda_1 = -\frac{\pi^2 EJ}{l^2}\frac{\sqrt{3}}{F}$

moltepl. 2

minimum λ_i in absolute value

$\lambda_2 = +\frac{\pi^2 EJ}{l^2}\frac{2\sqrt{3}}{F}$

minimum $\lambda_i > 0$

$$
\begin{aligned}
\delta U_{\mathrm{i}} &= \iiint_V \delta\,\underline{\epsilon}^\top \left(\underline{\sigma}_0 + \underline{\underline{\mathrm{D}}}\,\underline{\epsilon}\right) dV \\
&= \iiint_V \left[\underline{\underline{\mathrm{B}}}(\underline{\mathrm{d}})\delta\,\underline{\mathrm{d}}\right]^\top \left(\underline{\sigma}_0 + \underline{\underline{\mathrm{D}}}\,\underline{\underline{\mathrm{B}}}(\underline{\mathrm{d}})\,\underline{\mathrm{d}}\right) dV \\
&= \dots \\
&= \delta\,\underline{\mathrm{d}}\,\left(\left(\underline{\underline{\mathrm{K}}}_{\mathrm{ej}}^{\mathrm{M}} + \underline{\underline{\mathrm{K}}}_{\mathrm{ej}}^{\mathrm{G}}\right)\underline{\mathrm{d}} + \mathrm{o}(\underline{\mathrm{d}})\right).
\end{aligned}
$$

$$\left(\underline{\underline{\mathrm{K}}}^{\mathrm{M}} + \lambda\,\underline{\underline{\mathrm{K}}}^{\mathrm{G}}\right)\delta\,\underline{\mathrm{d}} = \delta\,\underline{\mathrm{F}} \qquad (211)$$

$$\left(\underline{\underline{\mathrm{K}}}^{\mathrm{M}} + \lambda_i\,\underline{\underline{\mathrm{K}}}^{\mathrm{G}}\right)\delta\,\hat{\underline{\mathrm{d}}}_i = 0 \qquad (212)$$

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