THE USE OF SOAP FILMS IN SOLVING TORSION PROBLEMS.

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Introduction.—The equations which represent the torsion of an elastic bar of any uniform cross-section are of exactly the same form as those which represent the displacement of a soap film, due to a slight pressure acting on its surface, the film being stretched over a hole in a flat plate, of the same shape as the cross-section of the bar. The theory of this relationship is briefly outlined, and it is shown that advantage may be taken of the analogy to find the stresses and torsional stiffness of a twisted bar or shaft of any cross-section whatever, by making appropriate measurements of soap films. The method is technically useful because there is no restriction on the shape of sections with which it is capable of dealing, whereas the number of cases in which the equations can be solved analytically is extremely limited.

The apparatus used for measuring films is described and illustrated, and examples of its use are given. These include simple geometrical figures, for which the results of the soap-film method may be checked by calculation, and also two instances of technically important sections which are not amenable to mathematical treatment. In the first of these, the magnitude of the stress in an internal corner, and its dependence on the radius of the fillet, are investigated, while in the second the stresses and torque of a twisted aeroplane wing spar, of I section, are discussed, and comparisons between the results of the method and those of some direct torsion experiments are given.

Finally, a number of general theorems relating to, and approximate formulæ for, the stiffness and strengths of shafts and beams, are obtained with the help of the soap-film analogy. It is shown by comparison with other results, that it is possible to deduce thus, in nearly every case, figures for those torsional data usually required in practice, which are within a few per cent. of the exact values. The superiority of these formulæ over those now in use appears to be due to the introduction, it is believed for the first time, of the length of the perimeter of the cross-section as a factor. This was suggested almost immediately by the soap-film analogy, and is an instance of the value of the latter as a means of forming a clear idea of the nature of the torsion problem.

General Considerations.—In the old theory of the torsion of shafts or beams of uniform cross-section, which was originated by Coulomb, it was assumed that sections of the bar, initially plane and at right angles to the axis of torsion, remained so when the bar was twisted, and that the only strains set up were those due to the relative rotation of adjacent sections about the axis.

In his classical memoir on the mathematical theory of torsion, Saint-Venant showed that the assumptions made by Coulomb were valid only in the case of circular shafts, either solid or having concentric circular holes. In every other instance the initially plane section is distorted into a curved surface and the stresses and strains set up in the bar cannot be calculated until the shape of this curved surface has been found.

A complete discussion of the theory of torsion put forward by Saint-Venant would be out of place in the present Paper. It is fully dealt with in books on the mathematical theory of elasticity, among which the treatise of Professor Love* may be mentioned. It

* A. E. H. Love. Mathematical Theory of Elasticity, 2nd. Edn. Chap. XIV.

is necessary to remark, however, that he showed how to reduce the problem to that of finding a function of the co-ordinates of points on the cross-section, which satisfies a certain partial differential equation. There is, however, no known general analytical method of finding this function for any assigned cross-section, and therefore the torsion problem cannot be solved mathematically for the great majority of technically important sections.

A simple method of determining these stresses would be of the very greatest assistance in general engineering work, and even more so in the many fresh problems which have to be dealt with in aeronautical calculations. In the very complex sections which occur in this work, such as those of airscrew blades, and the many forms of spars and struts, etc., used, it is of the highest importance that correct knowledge should be available, and therefore the Authors have carried out work at the Royal Aircraft Factory, Farnborough, with a view to solving the problem by means of a simple experimental method. The following is a very brief description of the method which has been developed.

A hole is cut, in a thin plate, of the section required to be investigated, and a circular hole of a predetermined diameter is cut alongside it. The plate is placed in a box and soap films are stretched across the holes. The films are blown out slightly by reducing the air-pressure on one side of them. By making suitable measurements of the shape of the resulting film surfaces, as will be explained later, it is possible to find the stresses in a bar of the given section, in terms of the stresses in a circular bar of the same diameter as the circular hole, when the two bars are twisted through the same angle per unit length. It is equally easy, by means of other measurements, to find the ratio of the torques which must be applied to the two bars, in order to produce the same twist in each. It will readily be seen that by this means the most complicated sections can be dealt with.

The experimental work is described in the body of the Paper, while the mathematical theory of the method is discussed in an Appendix.

Experimental Methods.—It is seen from the mathematical discussion given in the Appendix (page 785), that, in order that full advantage may be taken of the information on torsion which soap films are capable of furnishing, apparatus is required with which three kinds of measurements can be made, namely:—

(A) Measurements of the inclination of the film to the plane of the plate at any point, for the determination of stresses.

(B) Determination of the contour lines of the film.

(C) Comparison of the displaced volumes of the test film and circular standard for finding the corresponding torque ratio.

The available means of measurement will now be enumerated under these three heads.

(A) For this purpose optical reflection methods naturally suggest themselves. In the apparatus used by the Authors, the image of an electric lamp filament is viewed in the film in such a way that the reflected ray is coincident with the incident one, so that their common direction gives the inclination of the normal to the surface of the film. This experiment may conveniently be referred to as the measurement of angles by auto-collimation.

(B) For mapping contour lines, a steel needle point, moistened with soap solution, is arranged to move about over the plate carrying the film, its distance therefrom being adjustable by means of a micrometer screw. The point is made to approach the film till the distortion of the image in the latter shows that contact has occurred. This position is remarkably definite, so much so, indeed, that it is possible, with ordinary care, to limit the error in the measurement of the normal co-ordinate to $\pm 0 \cdot 001$ inch. This method of mapping contours will be referred to as the "spherometer" method. Another method, which was suggested to the Authors by Mr. Vernon Boys, F.R.S., though not so convenient as the one already described, is, nevertheless, useful in affording a ready means of exhibiting the shape of the contour lines to the eye. If a film be left undisturbed for, say, fifteen minutes, owing to drainage and consequent thinning of the film, a black spot appears at the highest point and gradually increases in size till, after the lapse of several hours, it may include the whole surface of the film. Its edge is quite sharply defined and is horizontal. Hence, if the plate has been levelled up beforehand, the edge of the black spot coincides at any moment with a contour line of the film.

(c) The most obvious way of measuring the displaced volume of the films is to blow them up by running a known volume of water, or, preferably, soap solution, into the apparatus from a pipette or burette. The volume of the circular film may be calculated from the observed value of the inclination at its boundary, since its surface is a portion of a sphere, and hence the volume of the other film may be obtained by difference. The most accurate results are obtained by giving the film a slight initial displacement before running in the known volume of liquid, and measuring the difference of the inclinations at the boundary of the circular film.

Another method, which requires a certain amount of practice, but which has the advantage of great simplicity, is to blow up the two films, observe the angle at the edge of the circular one, and then carefully place a flat plate, moistened with soap solution, on the test film, so as to cover it completely, until the flat plate is in contact with the test-plate. The total volume is then contained in the circular film, and it can be determined in the ordinary way by again observing the inclination. Hence the volume of the test film may be found.

Description of Apparatus.—In the apparatus used by the Authors, the films are formed on holes cut to the required shapes in flat aluminium plates, of No. 18 S.W.G. thickness. The plates are held in a horizontal position during the experiment, and the edges of the holes are chamfered off on the under side, to an angle of about 45°, in order to fix the plane of the boundary. The soap solution used is that recommended by Mr. Boys, namely, pure sodium or potassium oleate, glycerine and distilled water. It may be obtained ready for use from Messrs. Griffin, Kingsway, London.

The photograph, Plate 12, shows the apparatus in which the films are formed, and also illustrates the construction of the spherometer. The test-plate is clamped between the two halves of the cast-iron box A. The lower part of this box takes the form of a shallow tray $\frac{1}{4}$-inch deep, blackened inside and supported on levelling screws, while the upper portion is simply a square frame carrying the clamping studs and enamelled white inside. A three-way cock communicates with the former and a plain tube with the latter. The film shown in the photograph represents a section of an airscrew blade. It will be noticed that a black spot has commenced to form at the top of the bubble.

The spherometer apparatus consists of a screw B, of 1 mm. pitch passing through a hole in a sheet of plate glass $\frac{1}{8}$-inch thick and sufficiently large to cover the box in any possible position. It slides about on the flat upper face of the latter. The lower end of the screw carries a hard steel point C, tapering about 1 in 4, and its divided head moves beside a fixed vertical scale. Fixed above the screw and in its centre line is the steel recording point D. The record is made on a sheet of paper fixed to the board E, which can swing about a horizontal axis at the same height as D. To mark any position of the screw, it is merely necessary to prick a dot on the paper by bringing it down on the recording point.

In the auto-collimator, Plate 12, light from the straight filament of the 2-volt bulb A is reflected from the surface of the film through a V-nick B and a pin-hole eye-piece C, placed close to the lamp and shaded from direct light by a small screen. The inclinometer D, which measures the angle which the optical axis makes with the vertical, consists of a spirit level, of 6 feet radius, fixed to an arm which moves over a quadrant graduated in degrees. The apparatus is mounted, by means of a stiff-jointed link, on a tripod stand weighted with lead. Fine adjustment of angle is made with a screw.

Method of using Apparatus.—In using the soap film apparatus, the test-plate and lower half of the test-box, which must both be perfectly clean, are moistened with soap solution and clamped together by means of the upper frame. The soap solution not only forms an airtight joint between the plate and box, but also serves to saturate the air within the apparatus, so that evaporation from the surface of the film is minimized. The edges of the holes are now tested with the spherometer point; if they are not parallel to the plane of motion of the glass plate they must be adjusted. A film is then drawn across the holes by means of a strip of celluloid wetted with soap solution fresh from the stock bottle and the glass cover immediately replaced. The blowing up should be done by suction from the tube in the upper frame, and not by blowing through the stop-cock, as the carbon dioxide introduced by the latter method might affect the life of the film adversely. Measurements may now be made as desired. It should be remembered that if the auto-collimator is used, the apparatus must be levelled up beforehand.

In the case of the spherometer, the point, previously moistened with fresh solution, is set to a given height and made to touch the film at a number of positions, which are marked on the paper. This is repeated for as many contour lines as may be required. The plate need not be levelled. A contour map taken in this way is to be seen on the board, Plate 12.

Usually, the use of the auto-collimator is confined to the determination of inclinations at given points on the boundary, which are marked by scratches on the plate. It is better for stress measurements than the contour line method, since it gives the inclination directly, whereas in the other case the latter can only be found by a graphical differentiation. The use of the optical method may be extended to the finding of inclinations at points other than those on the boundary, with the help of the spherometer, in the following manner. The outline of the experimental hole is marked on the paper by means of the recording point, and the position of the point for which the stress is desired is added. The glass plate is adjusted until the recording point coincides with it. The needle is screwed down till it just touches the film, and its height is noted. It is then screwed back till the film breaks away and finally brought down again to within one or two thousandths of an inch of its former height. The auto-collimator is now adjusted till the image of the filament is seen in the film just below the needle-point. The reading of the inclinometer then gives the required angle.

Accuracy of Results.—Strictly speaking, the soap film surface can only be taken to represent the torsion function if its inclination γ is everywhere so small that $\sin \gamma = \tan \gamma$ to the required order of accuracy. This would mean, however, that the quantities measured would be so small as to render excessive experimental errors unavoidable. A compromise must therefore be effected. In point of fact, it has been found from experiments on sections for which the torsion function can be calculated, that the ratio of the stress at a point in any section to the stress at a point in a circular shaft, whose radius equals the value of $\frac{2A}{P}$ for the section, is given quite satisfactorily by the value of $\frac{\sin \gamma}{\sin \mu}$ where γ and μ are the respective inclinations of the corresponding films, even when γ is as much as 35°. Similarly, the volume ratio of the films has been found to be a sufficiently good approximation to the corresponding torque ratio, for a like amount of displacement.

In contour mapping, the greatest accuracy is obtained, with the apparatus at present in use, when μ is about 20°. That is to say, the displacement should be rather less than for the other two methods of experiment.

In all soap film measurements the experimental error is naturally greater the smaller the value of $\frac{2A}{P}$. Reliable results cannot be obtained, in general, if $\frac{2A}{P}$ is less than about half an inch, so that a shape such as a rolled I beam section could not be treated satisfactorily in an apparatus of convenient size. As a matter of fact, however, the shape of a symmetrical soap film is unaltered if it be divided by a septum or flat plate which passes through an axis of symmetry and is normal to the plane of the boundary. It is therefore only necessary to cut half the section in the test-plate and to place a normal septum of sheet metal at the line of division. This device, for the suggestion of which the Authors are indebted to the late Dr. C. V. Burton, may also be employed in many other cases where contour lines are so nearly normal to the septum that they are not sensibly altered by its introduction. An I beam, for instance, might be treated by dividing the web at a distance from the flange equal to two or three times the thickness of the web. It has been found advisable to carry the septum down through the hole so that it projects about $\frac{1}{8}$ inch below the underside of the plate, as, otherwise, solution collects in the corners and spoils the shape of the film.

The values set down in Table 1 (page 764), indicate the degree of accuracy obtainable with the auto-collimator in the determination of the maximum stresses in sections for which the torsion function is known. They also give an idea of the sizes of holes which have been found most convenient in practice. The angles given are (α) the maximum inclination at the edge of the test film, and (β) the inclination at the edge of the circular film of radius $\frac{2A}{P}$. They are usually the means of about five observations and are expressed in decimals of a degree.

The last two columns show the errors due to taking the ratio of angles and the ratio of sines respectively as giving the stress ratio.

The error is always positive for α/β, and its mean value is 1·98 per cent. In the case of $\frac{\sin\alpha}{\sin\beta}$ the average error is only 0·62 per cent. In only two instances does the error reach 1 per cent., and in both it is negative. The presence of sharp corners seems to introduce a negative error which is naturally greatest when the corners are nearest to the observation point. Otherwise, there is no evidence that the error depends to any great extent on the shape. Nos. 4, 5, 7 and 8 in the Table are examples of the application of the method of normal septa.

Table 2 (page 765) shows the results of volume determinations made on each of the sections 1–8 given in the previous Table.

TABLE 1.

Showing Experimental Error in determining Stress by means of Soap Films.

	Section.	Radius of Circle.	α.	β.	$\frac{\alpha}{\beta}$	$\frac{\sin \alpha}{\sin \beta}$	True Value.	Error $\frac{\alpha}{\beta}$	Error $\frac{\sin \alpha}{\sin \beta}$
		inches.	deg.	deg.				per cent.	per cent.
1	Equilateral Triangle: Height, 3 in.	1·00	32·55	21·19	1·536	1·490	1·500	+2·4	−0·7
2	Square: Side, 3 in. . . .	1·5	29·11	21·34	1·364	1·337	1·350	+1·0	−1·0
3	Ellipse: Semi-axes 2 in. × 1 in. .	1·296	30·71	24·32	1·263	1·240	1·234	+2·4	+0·5
4	Ellipse; 3 in. × 1 in. . . .	1·410	31·10	24·00	1·296	1·270	1·276	+1·6	−0·5
5	Ellipse: 4 in. × 0·8 in. . .	1·196	35·35	26·58	1·331	1·293	1·286	+3·5	+0·5
6	Rectangle: 4 in. × 2 in. . .	1·333	31·70	22·36	1·418	1·380	1·395	+1·6	−1·1
7	Rectangle: 8 in. × 2 in. . .	1·60	34·83	27·23	1·279	1·247	1·245	+2·7	+0·2
*8	Infinitely long rectangle: 1 in. wide	1·00	36·42	36·19	1·006	1·005	1·000	+0·6	+0·5

* On 4-inch length.

The average error is 0·89 per cent. In four of the eight cases considered the error is greater than 1 per cent., and in three of these it is negative. One may conclude that the probable error is

TABLE 2.—*Showing Experimental Error in determining Torques by means of Soap Films.*

No.	Section.	Maximum Inclination.	Observed Volume Ratio.	Calculated Torque Ratio.	Error.
		deg.			Per cent.
1	Equilateral Triangle: Height, 3 in.	32·06	1·953	1·985	−1·6
2	Square: Side, 3 in.	30·39	1·416	1·432	−1·1
3	Ellipse: Semi-axes, 2 in. × 1 in.	30·50	1·143	1·133	+0·9
4	Ellipse: 3 in. × 1 in.	31·01	2·147	2·147	0
5	Ellipse: 4 in. × 0·8 in.	36·12	3·041	3·020	+0·7
6	Rectangle: Sides, 4 in. × 2 in.	31·33	1·456	1·475	−1·3
7	Rectangle: 8 in. × 2 in.	35·28	1·749	1·744	+0·3
*8	Infinitely long rectangle	36·00	0·858	0·848	+1·2

* On 4-inch length.

somewhat greater than it is for the stress measurements, and that it tends to be negative. Its upper limit is probably not much in excess of 2 per cent. The remarks already made regarding the dependence of accuracy on the shape of the section apply equally to torque measurements.

As additional confirmation of the correctness of solutions of the torsion problem obtained by the soap film method, some experiments on wooden beams may be cited. In the first of these, a walnut plank was shaped so that its section was exactly the same as the hole in one of the test-plates, which represented a section of an air-screw blade, of fineness ratio 10·55, having its thickest part about a third of the way from the leading edge. The value of the modulus of rigidity, N, was found by performing a torsion test on this plank, using the expression for the torque given by a soap-film experiment on the plate which was used in shaping the plank. N was found to be $0 \cdot 1355 \times 10^6$ lb. per square inch. Five circular rods were then cut from the plank and their rigidities were measured. The mean value of N found in this way was $0 \cdot 1387 \times 10^6$, a difference of only $2 \cdot 3$ per cent.

Similar experiments were made on three lengths of spruce wing-spar, of I section. The results are set down below. Column A shows the value of N, obtained by twisting the spar, using the figure for torque obtained by soap films. Columns B and C show the values of N found from round specimens cut from the thickest part of the two flanges, while column D gives the percentage difference between A and the mean of B and C.

TABLE 3.—*Comparison of Soap Film Results with those of Direct Torsion Experiments.*

Spar No.	A. Lb. per square inch $\times 10^{-6}$.	B. Lb. per square inch $\times 10^{-6}$.	C. Lb. per square inch $\times 10^{-6}$.	D. ——
				Per cent.
1	0·1091	0·1172	0·1063	2·5
2	0·0873	0·0640	0·0966	8·7
3	0·1156	0·1200	0·1151	1·7

The comparatively large discrepancy in No. 2 is probably due to the extraordinarily large variation of N over this particular spar.

When contour lines have been mapped, the torque may be found from them by integration. If the graphical work is carefully done, the value found in this way is rather more accurate than the one obtained by the volumetric method. Contours may also be used to find stresses by differentiation, that is, by measuring the distance apart of the neighbouring contour lines; but here the comparison is decidedly in favour of the direct process, owing to the difficulties inseparable from graphical differentiation. The contour map is, nevertheless, a very useful means of showing the general nature of the stress distribution throughout the section in a clear and compact manner. The highly stressed parts show many lines bunched together, while few traverse the regions of low stress, and the direction of the resultant stress is shown by that of the contours at every point of the section. Furthermore, the map solves the torsion problem, not only for the boundary, but also for every section having the same shape as a contour line.

Experimental Results.—The two examples which follow serve to illustrate the use of the soap-film apparatus in solving typical problems in design:—

(1) It is well-known that the stress at a sharp internal corner of a twisted bar is infinite or, rather would be infinite if the elastic equations did not cease to hold when the stress becomes very high. If the internal corner is rounded off the stress is reduced; but so far no method has been devised by which the amount of reduction in strain due to a given amount of rounding can be estimated. This problem has been solved by the use of soap films.

An L-shaped hole was cut in a plate. Its arms were 5 inches long by 1 inch wide, and small pieces of sheet metal were fixed at each end, perpendicular to the shape of the hole, so as to form normal septa. The section was then practically equivalent to an angle with arms of infinite length. The radius in the internal corner was enlarged step by step, observations of the maximum inclination at the internal corner being taken on each occasion.

The inclination of the film at a point 3·5 inches from the corner was also observed, and was taken to represent the mean boundary stress in the arm, which is the same as the boundary stress at a point far from the corner, The ratio of the maximum stress at the internal corner to the mean stress in the arm was tabulated for each radius on the internal corner.

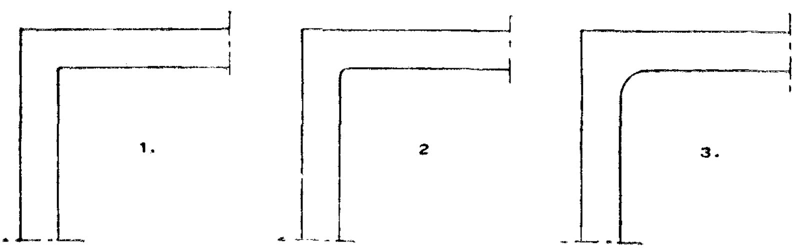
The results are given in Table 4:—

TABLE 4.—*Showing the Effect of Rounding the Internal Corner on the Strength of a twisted L-shaped Angle Beam.*

Radius of Internal Corner.	Ratio : $\frac{\text{Maximum Stress}}{\text{Stress in Arm}}$.
Inches.	
0·10	1·890
0·20	1·540
0·30	1·480
0·40	1·445
0·50	1·430
0·60	1·420
0·70	1·415
0·80	1·416
1·00	1·422
1·50	1·500
2·00	1·660

It will be seen that the maximum stress in the internal corner does not begin to increase to any great extent till the radius of the

FIG. 1.—*Stress in internal Corner.*



corner becomes less than one-fifth of the thickness of the arms. A curious point which will be noticed in connexion with the Table is the minimum value of the ratio of the maximum stress to the stress in the arm which occurs when the radius of the corner is about 0·7 of the thickness of the arm.

In Fig. 1 is shown a diagram representing the appearance of these sections of angle-irons.

No. 1 is the angle-iron for which the radius of the corner is one-tenth of the thickness of the arm. This angle is distinctly weak at the corner.

In No. 2 the radius is one-fifth of the thickness. This angle-iron is nearly as strong as it can be. Very little increase in strength is effected by rounding off the corner more than this. No. 3 is the angle with minimum ratio of stress in corner to stress in arm.

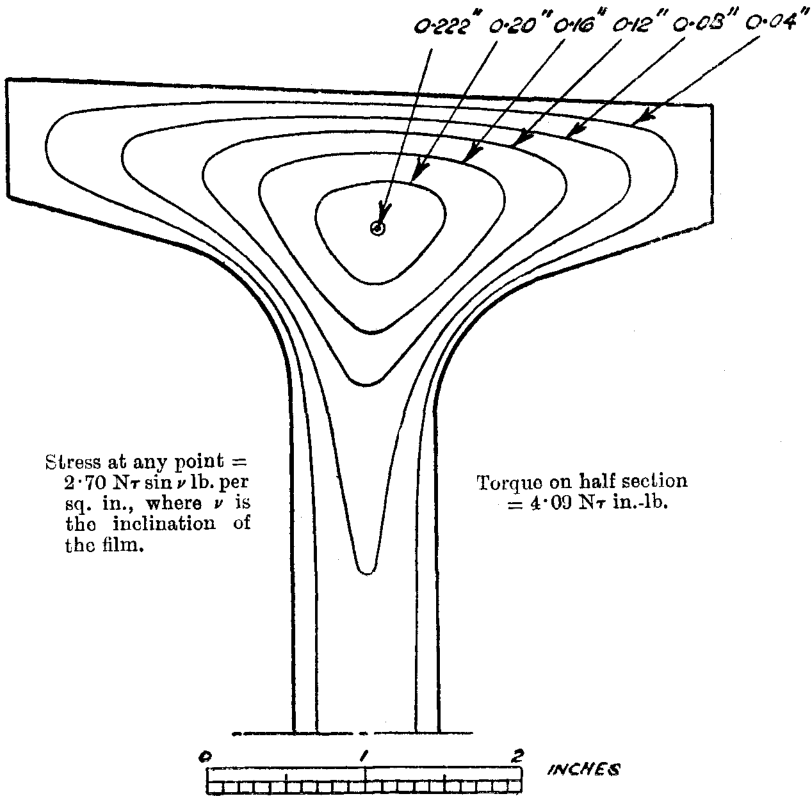
A further experiment was made to determine the extent of the region of high stress in angle-iron No. 1. For this purpose contour lines were mapped, and from these the slope of the bubble was found at a number of points on the line of symmetry of the angle-iron. Hence the stresses at these points were deduced. The results are given in the following Table.

TABLE 5.—*Showing the rate of Falling-off of the Stress in the Internal Corner of the Angle-Iron.*

Distance from Boundary.	Ratio: $\frac{\text{Stress at Point}}{\text{Boundary Stress in Arm}}$.
Inches.	
0·00	1·89
0·05	1·36
0·10	1·12
0·20	0·77
0·30	0·49
0·40	0·24
0·50	0·00

It will be seen that the stress falls off so rapidly that its maximum value is to all intents and purposes a matter of no

FIG. 2.—*Lines of Shearing Stress in the Torsion of a Wooden Spar to scale.*
The figures give the heights of the contour-lines of the corresponding soap film.



importance, if the material is capable of yielding. If the material is brittle and not ductile a crack would, of course, start at the point of maximum stress and penetrate the section.

(2) The diagram shown in Fig. 2, which represents the half section of a wooden wing spar, is a good example of the contour line method. The close grouping of the lines near the internal

radii, denoting high stress, is immediately evident. The projecting parts of the flange are lightly stressed and contribute little to the torsional stiffness. The stress at the middle of the upper face is, however, considerable, being in fact next in order of magnitude below that in the radii. The stress near the middle of the web is practically constant and equal to that in a very long rectangular section of the same thickness under the same twist.

A further point of interest is that the "unstressed fibre" is very near the centre of the largest circle which can be inscribed in the section. It will also be observed that the three points of greatest stress are almost coincident with the points of contact of the circle. The maximum stress is about 1·89 times the mean boundary stress.

The figures given below the diagram for the values of the stress and torque on the section fully confirm the generally accepted notions regarding the extreme weakness of I-beams in torsion.

General Deductions from the Soap Film Analogy.—One of the greatest advantages of considering the torsion problem from the soap film standpoint arises from the circumstance that it is very much easier to form a mental picture of a soap bubble than it is to visualize the complicated system of shear-strains in a twisted bar. It cannot be too strongly urged that the surest way of forming a clear idea of the nature of the torsion problem is to blow a few soap films on boundaries of various shapes. This can be done with the simplest of apparatus; the holes may be cut in plates of thin sheet metal, which can be luted on to the top of a biscuit tin with vaseline or soft soap. To blow the films up it is only necessary to bore a hole in the bottom of the tin and stand it in a vessel containing water. Two sections may readily be compared by cutting them in the same plate. A simple way of estimating inclinations is to view the image of the eye in the film and adjust the arm of a clinograph so that it lies along the line of sight. Black spots, as previously mentioned, may be observed if arrangements are made to cover the films with a sheet of glass, in order to exclude dust and air currents.

With the aid of simple apparatus of this kind the truth of theorems, such as those contained in the following list, may be readily demonstrated.

(*a*) The stress distribution (and therefore the torque) for any section is independent of the axis of twist. This is easily seen, since the shape of the soap film is completely determined by the boundary and the value of $\frac{4S}{p}$. Hence the torque on a number of bars clamped together at their ends may be found by adding the separate torques which would be necessary to twist each through the same angle. This, as in other cases, applies to torsion only. It will be realized that in practice there will be bending stresses which must be taken account of in the usual way.

(*b*) Any addition of material to a section must increase the torque, and *vice versâ*, so long as the distribution of material in the original section is unaltered.

(*c*) Any cut made in a section, whether it decreases the area or not, must decrease the torque.

(*d*) The stress at any point of the boundary of a section is never less than the boundary stress in a circular bar under the same twist, whose radius is equal to that of the circle inscribed in the section, which touches the boundary at the point in question.

More generally, if one section lie entirely inside another, so as to touch it at two or more points, the stresses in the inner figure are less than those in the outer one at the points of contact; if the two figures are approximately congruent in the neighbourhood of the points of contact, the difference between the stresses is small. The maximum stress in a section is not greater than $2aN\tau$, where a is the radius of the largest inscribed circle, unless the boundary is concave, that is, re-entrant.

(*e*) If a concave part of the boundary approximates to a sharp corner, the stress at this point may be very high, and if the curvature is infinite then the stress is also theoretically infinite, whatever be the situation of the corner with respect to the rest of the section. Actually, of course, if the material is ductile, we can only deduce that the stresses at such a corner are above the elastic limit.

(*f*) It is a consequence of (*e*) that it does not necessarily follow that the making of a cut in a section will reduce its strength, whether material is removed or not. As an example of this, one may quote the case of an angle-iron in which the internal corner is quite sharp. It is well known in practice that this will often fracture. It may be strengthened, however, by reducing the section, planing out a semicircular groove at the root of the angle-iron.

(*g*) There can be no discontinuous changes of stress anywhere in a section, excepting only those parts of the boundary where the curvature is infinite (concave or convex sharp corners).

(*h*) The maximum stress occurs at or near one of the points of contact of the largest inscribed circle, and not, in general, at the point of the boundary nearest the centroid, as has been hitherto assumed. An exception may occur if, at some other part of the boundary, the curvature is (algebraically) considerably less (that is, the boundary is more concave) than it is at this point.

(*i*) If a section which is long compared with its greatest thickness be bent so that its area and the length of its median line are unchanged, its torque will not be greatly changed thereby. For instance, the torsional stiffness of a metal plate is practically unaltered by folding or rolling it up into the form of an L or a split tube. Soap-film experiments show, in fact, that there is a diminution of less than 5 per cent. when the inner radius of the boundary is not less than the thickness at the bend.

(*j*) The "unstressed fibre," which is situated at the point corresponding with the maximum ordinate of the soap film, is near the centre of the largest circle which can be inscribed in the section.

In general, the inscribed circle has a maximum value wherever it touches the boundary at more than two points, and there is usually an unstressed fibre near the centre of each of these circles. Between each pair of maximum ordinates on the soap film, however, there is a "minimax" point, which is near the centre of the corresponding minimum inscribed circle. This fibre in the bar is also unstressed.

(*k*) The "lines of shearing stress" round the unstressed fibres of the first sort are initially ellipses, and round those of the second sort hyperbolæ, from which shapes they gradually approximate to that of the boundary. Notions of this sort are useful in practice, because it is possible, with their help, to sketch in the general nature of the lines of shearing stress for any section.

Approximate Formulæ for Torques and Stresses.—The torque on any section is given by

$$T = N\tau C$$

where C is a quantity of the fourth degree in the unit of length, which may be called the torsional stiffness of the section.

In the case of a circular shaft, in which there is no distortion of cross-sections, C is equal to the polar moment of inertia, so that we have

$$C = \tfrac{1}{2}Ar^2$$

where r is the radius of the circle.

In the general case we may put

$$C = \tfrac{1}{2}Ak^2$$

k is a length, which, by analogy with the circle, may be called the "equivalent torsional radius" of the section.

It is seen (*see* Appendix, page 787) that the mean stress round the boundary of any section is equal to the boundary stress in a circular shaft whose radius equals the quantity $\frac{2A}{P}$, which we have called h. This result suggested that some fairly simple approximate relation might be found between h and k.

When this idea was tested by application to known results, it became immediately evident that the fraction k/h was not very different from unity for a large number of sections. It was observed, however, that the presence of sharp outwardly projecting corners tended to make k greater than h, while the opposite effect was noticed in the case of sections whose length was great compared with their greatest thickness. For instance, h for the square is equal to a, the radius of the inscribed circle, whereas k is about

6 per cent. greater. In the equilateral triangle h is still equal to a, while k is 9 per cent. greater. For long rectangles and ellipses, however, k is considerably less than h.

At first sight, since, in many sections, these two effects are operating simultaneously, it might be thought that their separation, with a view to formulating a method of finding k empirically, would be a matter of some difficulty. It has been accomplished, however, by a process of successive trial, with the result that the empirical treatment about to be described has been evolved. The curves giving the values of the constants were found by plotting the values they should have for all the sections, for which a solution has been obtained, in order to get the correct result, and then drawing the best curves through these points.

If the figure contains sharp, outwardly projecting corners, construct a new figure by rounding off each corner with a radius r, which is a certain fraction of a, the radius of the largest inscribed circle. The value of this fraction depends on the angle θ, turned through by the tangent to the boundary in passing round the corner in question. In Fig. 3 (page 776), r/a is shown graphically as a function of θ/π, and, in addition, a Table of values is subjoined.

$\frac{\theta}{\pi}$	$\frac{r}{a}$	$\frac{\theta}{\pi}$	$\frac{r}{a}$
0·0	1·00	0·6	0·375
0·1	0·93	0·7	0·270
0·2	0·85	0·8	0·210
0·3	0·75	0·9	0·170
0·4	0·625	1·0	0·155
0·5	0·500		

If the area of this new figure be called A_1, and its perimeter P_1, the value of $\frac{2A_1}{P_1}$ is a close approximation to the k of the original boundary, subject to the second modification, which must be made for long sections.

It is not difficult to see that a certain amount of common sense may be required in applying the above rule. For instance, if the

FIG. 3.—Values of r/a in terms of θ/π.

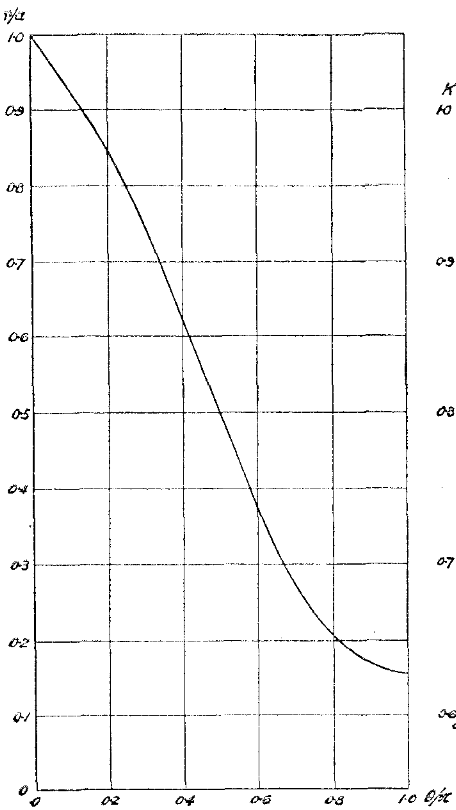


FIG. 4.—*Values of Torque Factor* K *in* $T = KN\tau \frac{2A_1^2}{P_1^2} A$.

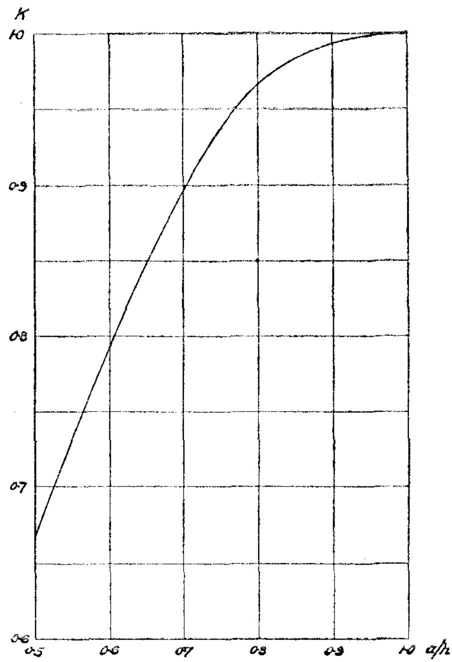


figure has a projection which is slightly rounded instead of being quite sharp, the value of r is that which would be used if the projection did run out to a sharp point. In most cases of this sort, however, it is found that the correction makes little difference.

The criterion, which has been adopted for fixing the value of the correction factor for long sections, is the fraction a/h. Where this is appreciably less than unity, the stiffness calculated by the process already described should be multiplied by the correction factor K, which is tabulated below, and which is also shown graphically in Fig. 4.

TABLE 6.

$\frac{a}{h}$	K	$\frac{a}{h}$	K
1·00	1·000	0·70	0·897
0·95	0·998	0·65	0·848
0·90	0·994	0·60	0·793
0·85	0·984	0·55	0·732
0·80	0·966	0·50	0·667
0·75	0·938		

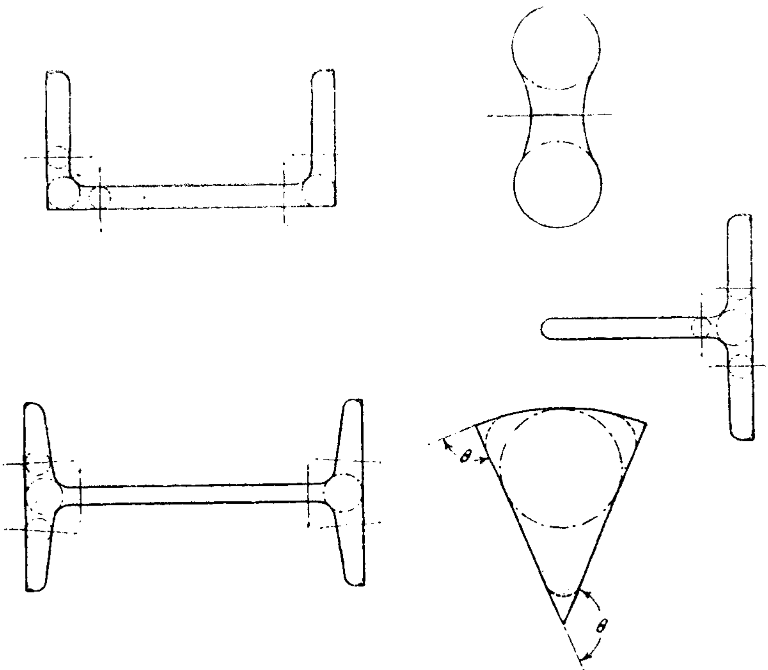
The expression for C now takes the form

$$C = \tfrac{1}{2} \text{K A} \left(\frac{2\text{A}_1}{\text{P}_1}\right)^2.$$

This formula is quite satisfactory for figures such as triangles, squares, ellipses, etc., in which a has one maximum value only, which may be called "simple sections," but if a has more than one maximum the solution in its present form is ambiguous, and it is necessary to split the section up into two or more parts, which will be referred to as the "components" of the original figure. The stiffness of each component must be found separately and the total stiffness obtained by addition.

In order to evolve a method of division, it is to be noted that the process already described is based on the equality of the resultant air-pressure and surface tension forces acting on the analogous soap film. If the film is divided by a series of "normal septa," which are so arranged that they are everywhere at right angles to the contour lines which they cut, the equilibrium equations are in no way altered, and the theorem is still true of each separate part of the film. Hence, if the section is divided in this way, the empirical treatment explained above should be applicable to each component. It is to be noted, however, that the term "perimeter" must be taken to mean that part of the boundary of the component which formed part of the perimeter of the

FIG. 5.—*Sub-division of Compound Sections.*



original figure. The remainder is not, strictly speaking, part of the boundary at all. It remains to formulate rules for the division of these "compound" sections.

Imagine a circle to be drawn in the section so as to *touch* the boundary at two points. Now let the centre of this circle move through the figure, the radius being varied simultaneously so that there is always contact at two points. At some places the circle will touch at three (or more) points. It is then an "inscribed circle of maximum radius," and between every such pair of maxima there must be a position where the radius is a minimum. The section should be divided by straight lines passing through the points of contact of these minimum circles.

In some cases, such as the web of an I-beam, there is a long thin parallel portion, and the position of the minimum circle is indeterminate. Here the line of division should be at a distance from the commencement of the parallel part equal to half its thickness. The portion of the web cut out may be treated separately as part of an infinitely long thin rectangle, the torsional properties of which are well-known. If the piece cut off is "closed" at the other end (*e.g.* the arm of an angle), it may be treated as a separate component. It is advisable to cut off all long, thin, projecting parts, in the same way, even though the sides are not quite parallel. The tapering flanges of I-beams may be cited in illustration.

Fig. 5 shows some typical examples of the sub-division of compound sections, and also illustrates the rounding off of sharp corners. The I-beam, for instance, has seven components, the channel five, and the tee four. In the 45° sector there is only one component. The angle turned through at the apex is 135°, so that $\theta/\pi = 0 \cdot 75$. Hence, from Table 5 (page 769), $r = 0 \cdot 24a$ (a is the radius of the chain-dotted circle). At the other two corners $\theta/\pi = 0 \cdot 5$, and hence $r = 0 \cdot 5a$.

In the case of certain sections, another form of expression may be arrived at, by a more direct method. Consider a soap film on a long narrow slit of varying width. If the rate of change of width with length is nowhere large, we may neglect the longitudinal curvature, $\frac{\delta^2 z}{\delta x^2}$, of the film, and put the transverse curvature $\frac{\delta^2 z}{\delta y^2}$, equal to a constant R, say, if the width at a distance x from one end be y, and the total length l, we readily obtain the volume, V, of the film in the form

$$V = \frac{1}{12R}\int_0^l y^3 dx,$$

3 I

This result must be exact for an indefinitely long rectangle, whence we have, by comparison with the known stiffness of the latter,

$$C = \frac{1}{3}\int_0^l y^3\,dx = \text{I, say,}$$

for the torsional stiffness of any long thin section.

A consideration of the case of ellipses suggests the modification

$$C = \frac{\text{I}}{1 + 4\dfrac{\text{I}}{\text{A}l^2}}$$

to allow for the longitudinal curvature of the figure.

The expression is now exact for all ellipses, whatever their fineness ratio, and, as will be seen, its error is within the limits of accuracy of soap film measurements, for sections such as those of airscrew blades, down to a fineness ratio of two, at least.

TABLE 7.

Section.	C (formula).	C (calculated).	Error.
			Per cent.
Square: Side $2s$	$2{\cdot}249\text{S}^4$	$2{\cdot}249s^4$	0
Rectangle: Sides, $2b$, $3b$ . .	$4{\cdot}710b^4$	$4{\cdot}698b^4$	$+0{\cdot}26$
,, ,, $2b$, $4b$ . .	$7{\cdot}320b^4$	$7{\cdot}318b^4$	$-0{\cdot}03$
,, ,, $2b$, $10b$ . .	$23{\cdot}15b^4$	$23{\cdot}31b^4$	$-0{\cdot}68$
,, ,, $2b$, $2l$. $l \rightarrow \infty$	$\frac{16}{3}lb^3$	$\frac{16}{3}lb^3$	0
Ellipse: Axes, $2b$, $3b$ . . .	$3{\cdot}250b^4$	$3{\cdot}260b^4$	$-0{\cdot}31$
,, ,, $2b$, $4b$ . . .	$5{\cdot}035b^4$	$5{\cdot}025b^4$	$+0{\cdot}20$
,, ,, $2b$, $10b$ . . .	$15{\cdot}14b^4$	$15{\cdot}10b^4$	$+0{\cdot}26$
,, ,, $2b$, $2l$. $l \rightarrow \infty$	$3{\cdot}235lb^3$	$\pi l{\cdot}b^3$	$+3{\cdot}00$
Equilateral Triangle: Side $2s$. .	$0{\cdot}3476\text{S}^4$	$0{\cdot}3464\text{S}^4$	$+0{\cdot}27$
45° Sector: Radius R . . .	$0{\cdot}01810\text{R}^4$	$0{\cdot}01815\text{R}^4$	$-0{\cdot}27$
90° ,, ,, R . . .	$0{\cdot}0830\text{R}^4$	$0{\cdot}0824\text{R}^4$	$+0{\cdot}73$
Curtate Sector: 180°, $\text{R}_1 = 2\text{R}_0$.	$1{\cdot}355\text{R}_0{}^4$	$1{\cdot}369\text{R}_0{}^4$	$-1{\cdot}02$

The formula may also be applied, though with somewhat less accuracy, to thin sections having a curved median line, provided that x is measured along the latter and y at right angles thereto.

Tables 7, 8 and 9 have been prepared to indicate the degrees of accuracy which may be expected in the application of the preceding formulæ.

TABLE 8.

Section.	C (formula).	C (soap film).
Wing Spar: $2\frac{1}{2}$ in. × $1\frac{1}{4}$ in. (I Section) .	0·0678 in^4	0·0680 in^4
,, ,, 3 in. × $1\frac{1}{2}$ in. ,, .	0·1042 in^4	0·1051 in^4
Angle: 3 in. × 3 in.	0·478 in^4	0·487 in^4
Airscrew Section: A	11·70 in^4	11·72 in^4
,, ,, B	7·44 in^4	7·50 in^4
,, ,, C	2·42 in^4	2·38 in^4
,, ,, D	0·846 in^4	0·835 in^4

TABLE 9.

Section.	C (formula).	C (experiment).
Angle: 1·175 in. × 1·175 in. .	0·01234 in^4	0·01284 in^4
,, 1·00 in. × 1·00 in. . .	0·00440 in^4	0·00455 in^4
Tee: 1·58 in. × 1·58 in. . .	0·01451 in^4	0·01481 in^4
I-Beam: 5·01 in. × 8·02 in. . .	1·160 in^4	1·140 in^4
,, 3·01 in. × 3·00 in. . .	0·1179 in^4	0·1082 in^4
,, 1·75 in. × 4·78 in. . .	0·0702 in^4	0·0635 in^4
Channel: 0·97 in. × 2·00 in. .	0·0175 in^4	0·0139 in^4

In Table 7, comparison is made with the results of Saint-Venant's exact analysis; in Table 8, the second column of values has been obtained from soap film measurements; while in Table 9 the results of the method are compared with those of some direct torsion experiments on rolled beams, carried out by Mr. E. G. Ritchie.*

It will be seen that all the figures in Table 9 show good agreement with the exception of those referring to the last three beams. In view of the remarks made by the Author cited in regard to the want of homogeneity of rolled beams, and more particularly the comparative weakness of the metal in the internal radii, the discrepancy in these cases cannot be considered unsatisfactory.

The method of calculating C should be chosen according to the nature of the section.

If there is only one maximum inscribed circle, and the section is not a long thin one, proceed by the method of rounding off sharp corners and finding $\frac{2A_1}{P_1}$, etc.

If the section is compound, divide it into its components and then proceed as before. Alternatively, if some of the components are thin compared with their length, they may be dealt with by finding $\int y^3\,dx$.

If the median line of the section is long in comparison with the greatest thickness, straighten out the median line where necessary and use the $\int y^3\,dx$ method.

Estimation of Stresses.—The empirical calculation of the stress at any given point of a section is naturally a matter of greater difficulty than the determination of torques. If the section contains no re-entrant angles, the stresses at the three points of contact of the inscribed circle of maximum radius a are usually given sufficiently well by the expression

$$\frac{2a}{1+m^2}\left[1+0\cdot15\left(m^2-\frac{a}{\rho}\right)\right],$$

* "A Study of the Circular Arc Bow Girder," by Gibson and Ritchie (Constable & Co., 1914).

where m is the quantity $\frac{\pi a^2}{A}$ and ρ is the radius of curvature of the boundary.

In the case of a "compound" section, the formula may be applied to each component separately.

The mean value of the stress round the boundary of any component is accurately equal to $2N\tau\frac{A}{P}$. By combining this value with those obtained for the maximum stresses, and bearing in mind the general properties of soap films, it is possible to sketch in a boundary stress diagram for the component, with sufficient accuracy for most purposes.

TABLE 10.

Section.	Stress $/N\tau$ (formula).	Stress $/N\tau$ (true).
Ellipse: Axes $2a$, $2b$	$\frac{2ab^2}{(a^2+b^2)}$	$\frac{2ab^2}{(a^2+b^2)}$
Square: Side $2s$	$1{\cdot}35s$	$1{\cdot}35s$
Rectangle: Sides $2s$, $3s$ . . .	$1{\cdot}64s$	$1{\cdot}69s$
,, ,, $2s$, $4s$ . . .	$1{\cdot}77s$	$1{\cdot}86s$
,, ,, $2s$, $8s$ . . .	$1{\cdot}94s$	$1{\cdot}99s$
,, ,, $2s$ . . .	$2{\cdot}00s$	$2{\cdot}00s$
Equilateral Triangle: Sides $2\sqrt{3}s$.	$1{\cdot}53s$	$1{\cdot}50s$
Wing Spar (I) $a = 1{\cdot}05$ in. . .	$2{\cdot}14$	$2{\cdot}13$*
,, ,, (I) $a = 1{\cdot}27$ in. . .	$2{\cdot}60$	$2{\cdot}58$*

* Determined by soap films.

Obviously the formula cannot be expected to apply to points where the boundary is concave—that is, re-entrant angles, since it fails to differentiate between an acute re-entrant angle and an obtuse one. It is possible to devise a formula which will take account of this angle and which will fit any assigned number of observed results within, say, four or five per cent., but such a formula naturally becomes more complicated as its range of application is increased, and hence the practical utility of such generalization is doubtful. Probably the most satisfactory way of dealing with re-entrant sections is to make soap film measurements and to deduce, from these, formulæ or curves which apply to one particular class of figure only.

It should be mentioned, however, that the formula given has been found to agree with soap film measurements on a number of re-entrant sections, in which the angle is approximately a right angle, when ρ is not very small. I-beams, channels and tees are examples of such sections, to which the formula may be applied. It should be borne in mind, however, that ρ is now negative.

The stress at any point of a rolled standard section may be taken to be $2aN\tau$, where a is the radius of the inscribed circle which touches at that point, except at places near the end of a flange, where the stress is smaller. The same thing holds for figures such as airscrew sections, when the fineness ratio is greater than about eight.

The Paper is illustrated by Plate 12 and 5 Figs. in the letterpress, and is accompanied by a Mathematical Appendix.

MATHEMATICAL APPENDIX.

The solution of the problem of torsion can be made to depend (*see* the book referred to in the introduction) on the finding of a function, ψ, of x and y, the co-ordinates of points on the cross-section, which satisfies the partial differential equation

$$\frac{\delta^2\psi}{\delta x^2} + \frac{\delta^2\psi}{\delta y^2} + 2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at all points of the cross-section, and is zero at all points on the bounding curve.

Consider the equations which represent the surface of a soap film stretched over a hole of the same size and shape as the cross-section of the twisted bar, cut in a flat plate, the film being slightly displaced from the plane of the plate by a small pressure p.

If S be the surface tension of the soap solution, the equation of the surface of the film is

$$\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} + \frac{p}{2S} = o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where z is the displacement of the film and x and y are the same as before. Round the boundary, of course, $z = o$.

It will be seen that if z is measured to such a scale that $\psi = \frac{4Sz}{p}$ then the two equations are identical. It appears, therefore, that the value of ψ corresponding with any values of x and y can be found by measuring the quantities p/S and z on the soap film.

To put the matter in another light, the soap film is a graphical representation of the function ψ for the given cross-section. Actual values of ψ can be obtained from it by multiplying the ordinates by $\frac{4S}{p}$.

If N is the modulus of rigidity of the material and τ the twist per unit length of the bar, the shear stress at any point of the cross-section can be found by multiplying the slope of the ψ surface at the point by $N\tau$, so that, if γ is the inclination of the bubble to the plane of the plate, the stress is

$$f_s = \frac{4S}{p} N\tau\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The torque T on the bar is given by

$$T = 2N\tau \iint \psi \, dx \, dy$$

$$\text{or } T = \frac{8S}{p} N\tau V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where V is the volume enclosed between the film surface and the plane of the plate.

The contour lines of the soap film in planes parallel to the plate correspond to the "lines of shearing stress" in the twisted bar, that is, they run parallel to the direction of the resultant shear stress at every point of the section.

It is evident that the torque on and stresses in a twisted bar of any section whatever may be obtained by measuring soap films in these respects.

In order to obtain quantitative results, it is necessary to find the value of $\frac{4S}{p}$ in each experiment. This might be done by measuring S and p directly, but a much simpler plan is to determine the curvature of a film, made with the same soap solution, stretched over a circular hole and subjected to the same pressure difference, p, between its two surfaces, as the test film.

The curvature of the circular film may be measured by observing the maximum inclination of the film to the plane of its boundary.

If this angle be called μ, then

$$\frac{4S}{p} = \frac{h}{\sin \mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where h is the radius of the circular boundary.

The most convenient way of ensuring that the two films shall be under the same pressure, is to make the circular hole in the same plate as the experimental hole.

It is evident that, since the two films have the same constant $\frac{4S}{p}$, we may, by comparing inclinations at any desired points, find the ratio of the stresses at the corresponding points of the cross-section of the bar under investigation to the stresses in a circular shaft of radius h under the same twist. Equally, we can find the ratio of the torques on the two bars by comparing the displaced volumes of the soap films. This is, in fact, the form which the investigations usually take.

As a matter of fact, the value of $\frac{4S}{p}$ can be found from the test-film itself by integrating γ, its inclination, round the boundary. If

A be the area of the cross-section, then the equilibrium of the film requires that

$$\int 2\mathrm{S}\, \sin\gamma\, ds = p\,\mathrm{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation may be written in the form

$$\frac{4\mathrm{S}}{\mathrm{P}} = 2 \times \frac{\text{area of cross-section}}{(\text{perimeter of cross-section}) \times (\text{mean value of } \sin\gamma)} \quad . \quad . \quad . \quad (7)$$

By measuring γ all round the boundary the mean value of $\sin\gamma$ can be found, and hence $\frac{4\mathrm{S}}{p}$ may be determined. This is, however, more laborious in practice than the use of the circular standard.

It is evident that if the radius of the circular hole be made equal to the value of $\frac{2\mathrm{A}}{\mathrm{P}}$, where A is the area and P the perimeter of the test hole, then $\sin\mu$ = mean value of $\sin\gamma$. It is convenient to choose the radius of the circular hole so that it satisfies this condition, in order that the quantities measured on the two films may be of the same order of magnitude.

The corresponding theorem in the torsion problem states that the mean stress round the boundary of a twisted bar is equal to the stress at the boundary of a circular shaft of radius $\frac{2\mathrm{A}}{\mathrm{P}}$. It is shown in the text that this property can be made the basis of a method of approximating to the torsional stiffness of any bar by calculation.

SYMBOLS AND FORMULÆ USED IN THE PAPER.

N = modulus of rigidity of material.
τ = twist of bar in radians per unit of length.
A = area of cross-section of bar.
P = length of perimeter of cross-section.
$h = \frac{2\mathrm{A}}{\mathrm{P}}$.
f_s = shear stress in bar.
f_c = shear stress in circular bar of radius h under twist τ.
T = torque applied to bar.

T_1 = torque applied to circular bar to give twist τ.
γ = inclination of soap film blown on a hole of the same shape as the twisted bar.
μ = inclination of film blown on a circular hole of radius h.
V = displaced volume of the test-film.
V_1 = displaced volume of the circular film.
S = surface tension of soap solution.
p = pressure difference causing displacement.
$C = \frac{T}{N\tau}$.
k = "equivalent torsional radius."
a = radius of inscribed circle.
r = radius for rounding projecting corners.
θ = angle turned through at a corner by the tangent to the boundary.
A_1 = area of modified section, when the corners have been rounded off.
P_1 = perimeter of modified section.
K = torque correction factor.
$I = \frac{1}{3}\int_0^l y^3\,dx$, the integration being taken along the median line of the section (l = length of median line).
$m = \frac{\pi a^2}{A}$.
ρ = radius of curvature of boundary of section.

(1) $f_s = \frac{4S}{p} N\tau\gamma$.

(2) $T = \frac{8S}{p} N\tau V$.

(3) $\frac{4S}{p} = \frac{h}{\sin\mu}$.

(4) $\frac{f_s}{f_c} = \frac{\sin\gamma}{\sin\mu}$, for any pair of points on the sections.

(5) $\frac{T}{T_1} = \frac{V}{V_1}$.

(6) $C = \frac{1}{2}KA\left(\frac{2A_1}{P_1}\right)^2$ for a simple section or for any component of a compound section.

(7) $C = \dfrac{I}{1 + \dfrac{4I}{Al^2}}$ for a long thin section.

(8) Stresses at points of contact of inscribed circles of maximum radius a

$$f_s = \frac{2aN\tau}{1 + m^2}\left[1 + 0{\cdot}15\left(m^2 - \frac{a}{\rho}\right)\right].$$

(9) Mean stress round the boundary of any section

$$f_s = \frac{2A}{P} N\tau.$$

(10) Stress at any point of the boundary of a rolled standard section

$$f_s = 2aN\tau$$

(a is the radius of the inscribed circle which touches at the point in question).

Discussion.

The PRESIDENT said the Paper was both original and very interesting. He did not know how many of those present had been able to follow it completely; he had found parts of it difficult, especially the method of dealing with the components of "compound sections." Then he was not quite clear whether the equations representing the displacement of a film by pressure really represented the surface of a soap-bubble. He thought these equations were based upon the assumption of a film of uniform thickness. The thickness of a soap bubble was not uniform, its lower part was thicker than its upper. Would not the variation of thickness alter the form of the bubble, and consequently of the inclinations of the tangent planes and of the volumes sufficiently to affect the strains and torsions derived from their measurement?

Mr. HENRY FOWLER, C.B.E., said he was glad to have an opportunity of speaking on the Paper, because it was in the factory of which he was at present Superintendent that this work had been

(Mr. Henry Fowler, C.B.E.)
carried out, and as Mr. Taylor had said, it was the result of the difficulties in which they found themselves owing to the impossibility of dealing mathematically with many of the problems met in aeronautics. He could not help feeling that, in solving these problems in the way the Authors had, they had conferred not only upon the aeronautical industry, but upon engineering generally a very considerable benefit. The Paper was undoubtedly somewhat difficult to follow to those who had left the drawing office some time ago, and therefore whose mathematics were something like the soap-bubble in that in course of time it got almost infinitely thin, but it was really so simple, and at the same time of such intense use that he hoped full advantage would be taken of the results in the Paper. The results were in an even more complex state when they were first placed before the Advisory Committee of the Aeronautics Committee—of which he regretted to see he was the only Member present. It was felt, however, that they were of such immense importance that it was very desirable they should be published at the earliest possible moment in the most convenient manner, and that they should be brought before the engineering profession in the best manner possible. He felt, and the Committee agreed, that that could not be better done than by asking for the results to be brought before a Meeting of the Institution of Mechanical Engineers.

The question, as Mr. Taylor had said, was only in its inception. The work which was being done and could be done with the soap-bubble was growing already, and Mr. Griffith with Mr. Taylor had obtained some other very remarkable results on problems which had been looked upon as impossible to deal with mathematically, and therefore which had been dealt with on the old method of strengthening up the portion of the structure which failed. It would be seen by the Paper—and especially if Mr. Griffith showed the members the apparatus—that it was perfectly simple to obtain the results with which the Paper dealt.

A point had been raised with regard to the difficulty of ascertaining accurately the contour lines. He was sorry that they had not a method by means of which this could be shown on the screen, but it would be found that they could depend upon getting an accuracy of one-thousandth of an inch and getting contact between the end of the pointers and the soap-bubble itself, and with care without breaking the bubble. Although that might seem very complicated, he was perfectly certain that anyone who had an opportunity of manipulating the apparatus would find that it was simplicity itself. In aeronautical work there were so many points in which the stresses were complex.

Mr. Taylor had spoken of the question of the air-screw or propeller. That was one instance where he had seen the greatest use made of this method. It was absolutely essential that there should be sufficient stiffness so that the blade should not deform. If it did, the result was that the pitch would be less. It was not in aeronautical work only, however, that the method had proved useful. Only that day he had been asked to carry out experiments with the method on certain engine crank-shafts and propeller-shafts, with which there had been trouble, and in connexion with which he felt perfectly certain that with methods such as this before him, the designer would have been able to carry out his work in a way which would have prevented the trouble arising. The method would have shown that it was only asking for trouble to get out designs in a certain way, and with the very simple method by which it was possible to look at the shape of the soap-bubble, and anticipate what the effect of torsion on the portion of the structure would be, one could at once reject certain shapes which otherwise would be gone on with.

It was only right to point out that Dr. Gibson, who was present, and Mr. Ritchie, both of whom were members of his staff, had dealt with the application of this method to structural work. In the case of a shop, for instance, where it was necessary to put a shafting on to a part of the roof members, torsional stresses were set up. If they synchronized, the tendency was to give considerable trouble. This method provided an opportunity of showing how the structure was likely to stand up against these stresses, and it could be arranged accordingly to deal with it before they absolutely encountered the trouble.

(Mr. Henry Fowler, C.B.E.)

One point which had been mentioned by Mr. Taylor would appeal very strongly to engineers, and that was the difficulty of the sharp corner. They all knew the way this was got over, but here they had a method of showing what they had to get over, and how much radius had to be put in, in order to get over it. Another point, and one which might seem somewhat startling, was the statement that metal might be taken away from a structure in order to make it stronger. In the particular case mentioned by Mr. Taylor, the section was made a great deal stronger, not by filling metal in, but by taking a portion out at the bottom of the section. In conclusion, he wished to express his appreciation of the method described in the Paper, and of the work of the Authors.

Professor A. H. GIBSON also expressed his appreciation of the importance of the Paper. There was no doubt that it was practically the only real advance in our knowledge of the general problem of torsion since the investigation of Saint-Venant. The Authors were mainly interested in aeronautics, and they seemed to think it was chiefly of importance as applied to aero work, but he regarded the Paper as even more important from the structural and purely mechanical engineering point of view. He was referring more particularly to the case mentioned by Mr. Fowler, of torsion in ordinary commercial rolled sections. All rolled sections were subject to some torsion in practice; even when every attempt was made to load a straight beam uniformly there was some torsion. When loads were applied eccentrically large torsion forces might be obtained, and when the problem was extended to the case of the beam which was not straight in plan, such as the beam supporting the circle of a theatre, supported at intervals and loaded uniformly in between, then the torsion was generally the all-important factor. He did not know whether engineers realized that, up to about three or four years ago, there was absolutely no data on which to base the design of structures made up of such sections, and subject to torsion, with any degree of exactitude. A few isolated experiments had been made in this country and in Germany, and about four years ago Mr. Ritchie, of Dundee, carried out an extensive series of tests on commercial sections—purely torsion tests—to determine the angular displacement in terms of the applied torque. That gave some useful data, and showed the excessive weakness of the majority of these sections under torsion, but did not enable us to determine the stresses in such a section, and, where it was necessary to obtain some idea of the stress, it was necessary to have recourse to the work of Bach and Auntenreith which was not particularly satisfactory. The work of the Authors had now made it possible to obtain, not only the angular twist corresponding to a given torque, but also the stress at every point of a section, and seeing that this could be done so easily and so cheaply, and realizing the importance of it, it was of the highest importance that the values for the majority of the commercial sections should be ascertained. He ventured to suggest that it was essentially a case where either the Engineering Standards Committee or the Institution Research Committee should consider the question of having the torsional coefficients of the various standard sections determined and tabulated, because it would be well worth the trouble expended in obtaining them.

As regards the Paper, there were one or two minor points to which he would like to make reference. The first had reference to Table 4 (page 768), which showed the effect of rounding the internal corner on the strength of an angle section. As most of them would anticipate, the section was distinctly strengthened at first by increasing the radius of the fillet at the inner corner, but it would seem from the Table that, after attaining a certain radius, about 0·7 times the thickness of the arm, any further increase in the radius caused the section to become distinctly weaker. That was how he read the Table, but he did not know if he was right in interpreting it in that way. It would be noticed that towards the end of the Table, as the radius became large, the rate of increase in the ratio of the maximum stress to the stress in the arm became very large, and it would be interesting if the work could be extended to even larger radii. He was not sure whether the Table meant that, in an angle subject to a given torque, the maximum stress increased as the radius increased beyond 0·7. At

(Professor A. H. Gibson.)
first sight this would appear improbable. Was it not the case really that the mean stress in the arm was reduced by the addition of material in the corner, so that in spite of the increase in the ratio given in column 2 of Table 4, the maximum stress in the section became smaller as the radius became greater. Perhaps Mr. Taylor would say if that was the right interpretation. The same thing appeared on page 784, where it was stated that "the stress at any point of a rolled standard section may be taken to be $2a\mathrm{N}r$, where a is the radius of the inscribed circle which touches at that point, except at places near the end of a flange, where the stress is smaller." That implied that if the size of the fillet between the web and the flange of an ordinary T section was increased, that increased the stress in that section under a given twist. Did that also apply under a given torque? The engineer wanted to know what the stress would be under a given torque, and not under a given twist. If the Table could possibly be modified in this direction, it would perhaps not be so likely to mislead the average engineer.

With reference to Table 7 (page 780), the experimental work in which was done by Mr. Ritchie, he (the speaker) had been working out the value of C by the formula given by the Authors on that page, and in the case of the first two angles and the last channel—each of which could be imagined to consist of a straight thin piece of metal bent round so as to form the section under consideration—results were obtained which were the same as those given by the Authors, and he inferred that that formula was the one used by the Authors in computing these values. In any case, it agreed quite well with the results, but when it came to sections 3, 4, 5 and 6, respectively T beams and I beams, the agreement was not close, and he would like to know exactly what formula the Authors used n those cases, and to what extent, if at all, they thought it possible to apply the simple formula at the top of page 780 to such sections.

Dr. L. BAIRSTOW, C.B.E., F.R.S., thought the experiments described by the Authors illustrated a growing tendency amongst engineers to take up problems left by mathematicians as insoluble.

In some cases the soap-film method of solving differential equations bore the same relation to the ordinary mathematical methods as the use of a slide-rule bore to the use of a book of logarithms, in that it gave a certain degree of approximation to a correct answer far more rapidly than the standard mathematical analysis. On the other hand, there were many cases where the problems had been given up by the mathematician, which could be solved by the use of soap films with an amount of labour scarcely greater than that in the mathematically simpler cases. The illustration given in the Paper of the solution of the differential equation for the torsion of cylinders was a particular case of a much more general problem, which could be solved by a combination of mathematical method and the use of a soap film, the general character of which could be illustrated by imagining a soap film on which the pressure, instead of being uniform, was variable from point to point. It was quite clear that there was no simple means of applying a non-uniform pressure to the soap film, but the difficulty could be removed by a standard piece of mathematical analysis. There were cases in which mathematical analysis alone was sufficient for a complete solution of a problem :—

For instance, in the case of the viscous fluid in the area behind a cylinder where eddies occurred, there was a distribution of vorticity, which when known sufficed to determine the stream-lines, if the general soap-film problem just mentioned was capable of solution ; it was not in this case necessary to have recourse to a soap film, although part of the problem might be attacked by that means. The method shown by the Authors of solving particular cases of the general differential equation was not unique. Most members of the Institution would remember the diagrams of fluid motion obtained by Dr. Hele-Shaw ; glycerine was forced between two pieces of plate-glass and coloured streaks suitably introduced were projected on a screen and mapped. The solution obtained in that way could also have been obtained by the soap film, but the reverse was not true, as all the problems to which the soap-film method was applicable could not be solved by Dr. Hele-Shaw's method, as the latter applied only to the case of zero pressure on

(Mr. L. Bairstow, C.B.E.)
the soap film together with a plane bounding contour. To get the feature of the soap film which corresponded with the stream-line problems of Dr. Hele-Shaw, it was necessary to interpose an obstacle into the centre of a uniformly inclined soap film, the plane of the obstacle being horizontal. For other problems, such as the eddies mentioned above, it was necessary to build up a special boundary instead of the plane one used to illustrate inviscid fluid motion. A similar problem arose and had been solved by the Authors for the flexure of beams. If the soap film were looked upon as an engineering device for solving equations, which would get rid of much calculation by spherical harmonics and Bessel functions, all of which were involved in the calculation of such cases as the mathematician could tackle, its value was clear and considerable.

There was one piece of mathematical analysis which the Authors appeared to have missed when they stated that the twisting of a propeller-blade was not amenable to mathematical analysis. It was quite true that most mathematical methods applied to spheres or cylinders, but a method had been found by M. Marcel Brillouin by which sections not having a simple symmetrical form could be subjected to mathematical analysis. The method appeared to be not yet fully proved, but was sufficiently good to warrant further attention. The number of problems awaiting engineering solution was very great, and it was a pleasure to find that methods of experimental analysis applied to difficult problems in mathematical physics were now coming to the fore.

Dr. H. S. HELE-SHAW, F.R.S. (Member of Council), thought there had probably been nothing so interesting in the progress of science as the methods which had in recent years thrown light in many ways on the relation between the particles of matter in problems which formerly seemed hopelessly obscure; for instance, the wonderful researches of photomicrography and what they had shown about the behaviour of materials under stresses exceeding the elastic limit, and the beautiful experiments of Professor Coker who used rays of polarized light through sheets of xylonite or celluloid for studying molecular effects of stress within the elastic limit. As the last speaker had mentioned the question of films for investigating the behaviour of particles in stream-line flow between sheets of glass, he would remark that these particles were so extraordinarily small that it seemed highly improbable we should ever be able to see them individually. This might be illustrated by the fact that to fill a glass (that is half-a-pint) at the rate of a million particles a second would take 47 million years. To-night the use of films had been shown to lead in an extraordinary way to a knowledge of stress between molecules, and it was an illustration of how completely opposite the methods of approach might be to a discovery. In the former case of the use of these films it was found that the behaviour of stream colour-bands flowing with clear glycerine gave apparently similar lines to those of the lines of flow past cylinders and ellipses which had been predicted as a result of mathematical investigation by Poisson, Kelvin, and Clerk Maxwell. These investigations had only been applied mathematically for the case of a perfect liquid which of course had no practical existence, but when compared with the flow obtained with a thin film of viscous liquid they were found to agree exactly. Then Sir George Gabriel Stokes, one of the greatest of mathematicians, proved that under the new experimental conditions the results must be true.* That was a case of assuming a result, and then by investigation finding out that it was so. In the case of the Authors' new discovery brought before the world for the first time to-night, this could not have happened. Nobody could possibly have foreseen that the contour of a soap bubble, hanging on an orifice, represented by the curvature of its surface the rate of variation of stress in a body under torsion. Even a mathematician like Mr. Taylor could scarcely have imagined that such would be the case. But what happened? The differential equation for torsion was found to be, with certain substitutions, exactly the same as the equation for the surface form of a soap-bubble film. Enormous credit was due to the mathematician whose imagination enabled him to see this relation, and Mr. Taylor was most modest in his remarks. He

* British Association Report, 1898.

(Dr. H. S. Hele-Shaw, F.R.S.)
had heard Mr. Taylor at the Royal Society reading Papers, and it was known that he was one of our coming mathematicians. He had shown his modesty by saying that a greater discovery than his was the discovery of Mr. Griffith, who had devised the wonderful apparatus by which the soap-bubble surfaces could be measured and plotted.

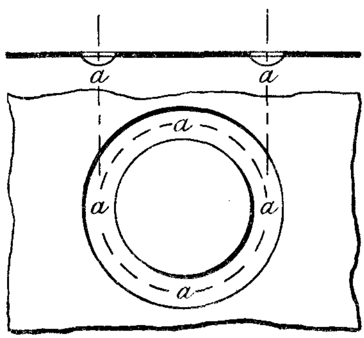
Engineers would know that it was one thing to see a mathematical relation and quite another to be able to work out results of practical value correctly, and still another to devise methods by which the properties of the hanging soap-bubble could be translated to solve engineering problems as the Authors had done. It was a most creditable thing to devise the beautiful piece of apparatus to measure the contour lines and plot them. At present it was impossible, as with Professor Coker's method, to see those lines on the screen, but perhaps a method would be found.

Now, as to the results, it was well known that in the case of a hill, if the contour lines were close together it indicated steepness, and thus having found the contour lines of the soap-bubble surface, the intensity of stress could not only be indicated but actually measured. Again, it was no small discovery to have found that the total torque was given by the actual volume and to measure that volume. The President's question was partly answered by the fact that actual result of the known case of a circular shaft was compared at the same moment with a film from the new section to be investigated, because they compared at the same time two films from the same material, and they were able by measuring both to compare what we knew about circular films with what we did not know about irregular shapes. He felt it was due to the Authors that a Member of the Council should express the view that, in having the Paper of to-night brought before the Institution in the first instance, a great compliment had been paid to it. It had not been brought before the Royal Society and then served up again to the Institution in a form supposed to be more suitable to the level of their intelligence. His own mathematics, like Mr. Fowler's, were naturally a little rusty, but he had read the Paper through from beginning to end and the mathematical explanations were quite clear, so that anyone with any knowledge of the calculus at all could follow it.

The speaker next alluded to some of the eleven general deductions on pages 772–3, each one of which dealt with one or other point of interest to engineers, and would each one no doubt form the subject of much future investigation. There was however one point which it seemed to him demanded some further explanation from the Authors:—In the Mathematical Appendix (page 785), equation 3 gave the stress at any point in terms of surface tension, pressure, modulus of rigidity, twist, and the angle of inclination at that particular point. This equation seemed directly applicable and quite correct as applied to a solid shaft of circular cross-section, and a limiting case, at which the angle of inclination (γ) was zero, was at the centre of the section where the stress was of course also zero.

When however the method was applied to a hollow circular shaft, it appeared to him to fail because the shape of the pendant film was as shown in Fig. 6. At the points *a a* on the sketch, and

FIG. 6.
Section of Hollow Shaft.
$\gamma = o$ at points *a a*.



round the dotted circular line, the angle of inclination (γ) was obviously zero, whereas the torsional stress at that point was clearly not zero.

Mr. TAYLOR said he would have to explain that a little more.

Dr. HELE-SHAW said this was only one of the points which might be discussed, but he was not now a professor or he would have derived great pleasure from following them up with students, as they appealed to him as throwing much light on the subject, and which to most engineers was not at all clear. He concluded by expressing the gratification they all felt that a work of high mathematical research, with such valuable practical applications, would be known all over the world as a British discovery.

Mr. JAMES J. GUEST thanked the Institution for the invitation to himself and several of his students at University College, London, to be present at the Meeting. The Paper had been most interesting to all of them. The mathematician occasionally made most important discoveries. For instance, Professor Clerk Maxwell made the startling discovery of the connexion of electricity and the ether and the ratio of the two constants—but it took many years before any direct experimental results confirmed his theory. In this case we were up to date; he did not know how long ago it was since Mr. Taylor introduced this analogy and experimental method, but it had come at a critical time. It was an exceedingly important Paper and must be taken in conjunction with the optical method of stress determination, which was most suitable for certain cases. It solved, experimentally, certain problems in bending which could not be tackled adequately by mathematics. The optical method however could hardly deal with torsion problems. Engineers, as a body, looked upon stresses and strains as partly direct, partly due to bending and partly torsional, and considered the latter pair as producing the more important effects.

The Authors had now given a method of determining, with fair accuracy, the torsional stresses produced by a torque, and as we previously possessed the optical method, the means of experimentally solving problems of the stresses produced in these simple cases were available. When however the problem became an irreducible one of three dimensions, these methods were not applicable.

With regard to the Paper, the question that was previously raised about the effect of the radius in the corner of the angle section gave him also some disquietude, and he would like to know the length of the legs of the angle in this particular case. This reduced to the same question which was asked earlier, but put the matter rather differently.

On page 770 there was a statement with which he disagreed namely: "It will be seen that the stress falls off so rapidly that its maximum value is to all intents and purposes a matter of no importance, if the material is capable of yielding." That, he thought, should be qualified. If the material were stressed in one direction occasionally the statement was true. But if the stress were an alternating stress, it was not true; failure would occur.

The propositions on page 772 did not appear to him to be put very clearly. Section (*a*) did not seem to be a stress-distribution question at all—it was merely a statement of the fact that the axis of a torque or bending moment could be moved about, provided it were kept parallel to itself. In (*b*) there was the statement "any addition of material to a section must increase the torque, and *vice versa*." He did not consider this true if a re-entrant angle was formed by the added material. For example, take a bar of circular section and add to it a strip round its circumference, completely encircling it except for a small part forming a sharp re-entrant angle. The stress produced there would be very high and the possible torque would be lowered. This was stated in section (*e*)—that the stress would be high, and therefore in the former case the addition of material weakened the shaft. Further, it was well known that material could be cut out of some structures, with a resulting increase to their strength. Probably the simplest case was Emerson's paradox. If a beam of triangular section were set with a face horizontal and loaded, the highest stress would occur at the apex. If, however, about one-sixth of the height were planed off from the vertex, there would be less stress in the beam under the same loading. On page 774, the meaning of the expression $\frac{1}{2}Ak^2$ might be confused with an established use of these letters.

Mr. G. I. Taylor, replying to the President's question as to

(Mr. G. I. Taylor.)

whether the fact that the film had not uniform thickness made any difference, said that the shape of the film depended on surface tension, and not to any appreciable extent on its weight. Variations in the thickness, while they altered the weight of the film, did not affect its surface tension. This had been pointed out by Professor Boys, who had made a special study of soap films. It was evident that the soap film would immediately break down if any considerable variation of surface tension over the film were possible, because the components of tension along the film would not then balance.

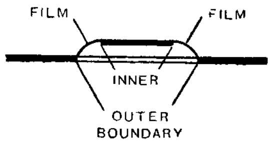
He had nothing particular to add to what Mr. Fowler had said, except that he considered that he and Mr. Griffith were particularly fortunate in finding an enlightened man like Mr. Fowler, who had given them the opportunity for carrying out this work.

With regard to what Professor Gibson said about Table 4 (page 768), he thought Mr. Griffith would agree with him that it was perhaps a trifle misleading. The assertion that the stress was a minimum for a given radius of the internal corner was actually true, whether the twist or the torque were constant, because Table 4 applied to L-shaped sections with very long arms, so that the additional torque due to the increased amount of metal in the corner was only a small portion of the total torque. In the case of an L section with shorter arms, the conclusions arrived at were probably very nearly true for a given twist, but not for a given torque. Mr. Guest had asked how long the arms of the L section were. They were practically infinite. It was not possible to obtain an infinite hole to stretch the bubble on, but, by putting a normal septum at the end of each of the arms, it was possible to obtain a film which was indistinguishable from the central portion of a film stretched over an infinite hole. In regard to the other questions raised by Professor Gibson, he would like Mr. Griffith to answer those relating to pages 780–1 because he did that work on the making of the empirical formulæ, and he really knew more about it.

It was very interesting to hear from Mr. Bairstow (page 796) that M. Marcel Brillouin had calculated the torsion function for an unsymmetrical section, but of course, only certain special unsymmetrical sections had been calculated, and he did not suppose that Mr. Bairstow meant to suggest that any given unsymmetrical section could be calculated. He was not aware of that before, however, and it was a very interesting fact.

With regard to Dr. Hele-Shaw's remarks about the twist of a hollow shaft (page 799), there they touched on quite new ground. The torsion of hollow shafts could not be done directly by this method. The Authors had solved the problem, but it was one of the many branches of the work which they had not yet published. In the case of hollow shafts, the film which represented the torsional stresses lay between an inner and an outer boundary as suggested by Dr. Hele-Shaw; but the two boundaries did not lie in the same plane. A method had been devised for finding out how

FIG. 7.



far apart the planes of the two boundaries must be, in order that the bubble which lay between them might be of the right shape. Fig. 7 showed a section of the two boundaries and the film. The shear stresses in a twisted hollow circular shaft were as might be expected greatest on the outside and gradually decreased till the inner surface was reached.

With regard to Dr. Hele-Shaw's remarks about the originality of the work, the Authors would like to repeat that the credit for the purely mathematical side of the work really belonged to Professor Prandtl, who pointed out in the *Physicalische Zeitschrift* for 1903 that the equations for the deformation of an evenly loaded membrane and the equations of torsion were identical in form.

Coming to the questions of Mr. Guest, he had already dealt with the length of the arms of the L section. The actual twisting of materials, upon which Mr. Guest had done a good deal of work lately, fell outside the scope of the Paper, but he was interested to

(Mr. G. I. Taylor.)
hear that, in the case of an alternating stress, the giving of the material was not of so much importance. This tended to increase the usefulness of the soap film method. He did not quite understand Mr. Guest's objection to the general remarks on page 772. The Paper did not say that an increase of material decreased the maximum stress, but that it necessarily increased the torque for a given twist. That is to say, it gave greater stiffness, but did not necessarily diminish the liability to breakage.

The PRESIDENT invited Mr. Griffith to send in his remarks in writing. In asking the members to give the Authors a hearty vote of thanks, he said he agreed with Dr. Hele-Shaw that the reading of the Paper was a great compliment to the Institution.

The vote of thanks was heartily accorded.

Mr. A. A. GRIFFITH, in a written reply to the Discussion, said the President had referred to the difficulties of the method of calculating the stiffness of a "compound" section, by dividing it up into its components. These difficulties undoubtedly existed, and the only royal road to success in overcoming them was to look at a number of soap films on holes of various shapes. It was then possible to visualize the film surface with fair accuracy in any particular case, and by keeping in view the general principle, that the lines of division should cut the contours more or less at right angles, one could not easily go wrong. He might refer here to Professor Gibson's difficulty in the application of the $\int y^3 dx$ formula to standard sections. Here again the solution was to be found in the assumption made in deducing the formula, namely, that "the rate of change of width with length is nowhere large." At the junction of the web and flange of an "I" or "Tee" beam the rate of change of width with length was not merely large, it was infinite, and he thought that Professor Gibson would agree that it was hardly fair to expect the formula to apply in such a case. For the same reason it could not be used for short rectangles, on account of the infinitely rapid change of thickness at their ends. The Authors used it chiefly for airscrew sections, for which it was exceedingly accurate and convenient.

There seemed to be a slight misconception in regard to the nature of the results yielded by the soap-film method. In any torsion problem there were three variables: torque, twist, and stress. If an engineer was concerned only with the strength of his structure, it was perhaps natural that he should consider only the maximum stress produced by a given applied torque and neglect the third quantity, twist. The soap-film method, however, essentially gave the torque and stress in terms of the angle of twist. If the relation between *torque* and stress were required, it must be found by eliminating τ from the other two expressions. This, he thought, explained the points raised by Professor Gibson and Mr. Guest in regard to the stress in the angle section and the general propositions on pages 772–3.

He was greatly interested in Mr. Bairstow's statement that mathematical solutions had been obtained for unsymmetrical shapes of airscrew section, but a great advantage of the experimental method was its quickness. It was easy to obtain the complete contour map of a section in less than an hour after the plate had been cut. In the mathematical method it was usually necessary to sum infinite series, whose convergence was often painfully slow, and they had learnt by experience that it was generally better to use the soap-film method in practical problems even when the solution in series was known.

Communications.

Mr. H. Carrington (Manchester) wrote, that in the Appendix (page 785) it was pointed out that the differential equation of the surfaces of the distorted cross-section of a twisted elastic prism and of the surface of the corresponding soap film were identical.

(Mr. H. Carrington.)

It was also stated that the value of ψ in equation 1 (page 784) was zero at all points of the bounding curve. In the case of a twisted elastic prism of uniform cross-section other than circular, the value of ψ was not in general zero, and the solution of equation 1 was correct only if ψ satisfied certain boundary conditions.* Since z in equation 2 (page 785), which was proportional to ψ in equation 1, was zero at all points of the boundary, it would appear that the solutions of the equation for the distorted section and the film were not in general identical.

The difference in the shape of the surfaces was also evident if the cases of twisted elastic prisms of triangular square or elliptic cross-section were considered. From the diagrams of Saint-Venant,* it was seen that all the contour lines of these three sections intersected the bounding curve, but for the corresponding soap film over a hole in a flat plate the contour lines were closed curves, and hence none of them intersected the edges of the hole.

Further, the stress at any point of the cross-section of a twisted elastic prism of cross-section other than circular was proportional to the sum of two quantities,* that is, to the angle of twist × the distance of the point from the axis of torsion + a quantity proportional to the slope of the distorted cross-section at the point. It was stated in the Paper that the stress was proportional to the angle of twist × the slope of the corresponding soap film at the point, but it was not explained how this single quantity was to be identified with the sum of the quantities giving the true stress at the point. It was, however, evident that if this single quantity represented the stress at the point, the film could not be a graphical representation of the distorted cross-section.

Again, in equation 4 (page 785) the torque was given by

$$\mathrm{T} = \frac{8\mathrm{S}}{p}\mathrm{N}\tau\mathrm{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

for a spherical film

$$\frac{\mathrm{S}}{p} = \frac{\mathrm{R}}{2}\dagger$$

* Thompson and Tait's Nat. Phil., Arts. 700–710.

† Lamb's "Statics," page 273.

Hence

$$T = 4R\,N\tau V,$$
$$= N\tau\,(4RV),$$

and for a circular shaft

$$T = N\tau\left(\frac{\pi}{2}r^4\right) \quad . \quad . \quad . \quad . \quad . \quad (b),$$

and according to the Paper 4RV should $= \frac{\pi}{2}r^4$. If h be the height, of the film above the plate

$$\frac{4RV}{\frac{\pi}{2}r^4} = \frac{4R\pi h^2\,(R-\frac{1}{3}h)}{\frac{\pi}{2}h^2\,(2R-h)^2}.$$

Equations a and b were certainly not identical.

When the maximum inclination of the film to the plate = 20°, and r = 2 inches, this ratio = about 2·1. Professor Gerald Stoney had also pointed out to the writer that, from a practical point of view, it was very improbable that there would be an unstressed fibre in a twisted I beam near the centre of the flange, as shown in Fig. 2 (page 770).

Dr. ALFRED W. PORTER wrote, that the method devised by the Authors was exceedingly ingenious and elegant, and should prove of great value in examining shearing stresses. It occurred to the writer, however, that a warning might be necessary in applying it to critical cases where the stresses became very large so as to approach the elastic limit. Both the equations employed, which were analogous to one another, were true only for small displacements. This suggestion was only put forward by way of caution, and was not intended to throw any doubt on the method when used within bounds. It must prove to be of great value and interest.

Mr. G. I. TAYLOR wrote, in reply, that Mr. Carrington had evidently not read the Appendix very carefully. It was nowhere stated that "the differential equation of the surfaces of the distorted cross-section of a twisted elastic prism and of the surface of the corresponding film were identical." Such a statement would

(Mr. G. I. Taylor.)
obviously be absurd. In Thomson and Tait's notation, the function γ, which represented the distortion of the cross-sections of the prism, was related to our function, ψ, by means of the equation

$$\psi = u - \tfrac{1}{2}(x^2 + y^2)$$

where u was the function conjugate to ψ, so that

$$\frac{d\gamma}{dx} = \frac{du}{dy} \text{ and } \frac{d\gamma}{dy} = -\frac{du}{dx}.$$

The two portions into which the stress at any point of the cross-section of a twisted prism were divided were both included in the function ψ. The part which depended on the angle of twist $\times$, the distance of the point from the axis of torsion, was derived from the portion $\frac{1}{2}(x^2 + y^2)$; while the part which was proportional to the slope of the distorted curve was derived from the other portion, u, of the function ψ. For fuller discussion of this, Mr. Carrington might refer to Love's "Mathematical Theory of Elasticity," Chapter XIV.

In regard to Mr. Carrington's remarks about circular shafts, the formula $\frac{s}{p} = \frac{R}{2}$ which he quoted from Lamb's Statics applied to a single surface of separation between the two media. A soap film had two surfaces; the connexion between the pressure difference on the two sides of a spherical form and its radius was accordingly $\frac{s}{p} = \frac{R}{4}$.

As was stated in the Appendix, the equations referred to a film slightly displaced from a plane. For this reason h was small and R large compared with the diameter of the hole; and *a fortiori* h was therefore small compared with R, and might be neglected.

Bearing these two points in mind, it would be seen that $T = \frac{8\,s}{p}(N\tau)\,V = N\tau\left(\frac{\pi}{2}r^4\right)$

Mr. A. A. GRIFFITH wrote that Mr. Carrington appeared to be under a misapprehension as to the nature of the function ψ used in demonstrating the soap film analogy. If the distortion of the cross-section (which he erroneously thought was the function ψ

referred to in the Paper) were called ϕ_1, then ϕ_1 was a plane harmonic function of the co-ordinates x and y within the section, and it satisfied a certain condition at the edge. It was known that for every such function there must exist a conjugate function ϕ_2, which was also a plane harmonic function, and whose slopes were connected with those of ϕ_1, by well-known relations. The condition satisfied by this second function at points on the boundary was found to be

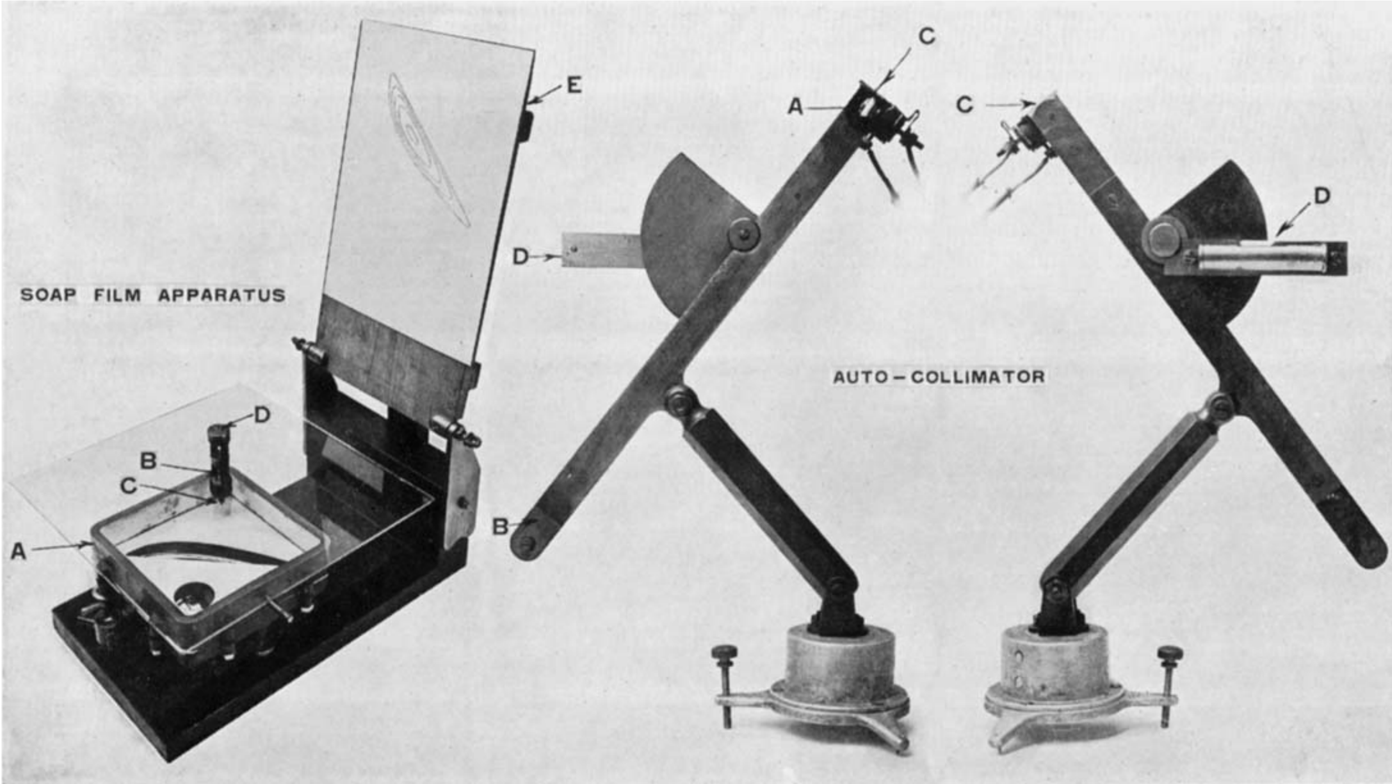
$$\phi_2 = \tfrac{1}{2}(x^2 + y^2).$$

The relation between ϕ_2 and ψ was given by

$$\psi = \phi_2 - \tfrac{1}{2}(x^2 + y^2),$$

and it was easy to see that ψ must satisfy the boundary condition $\psi = o$. If Mr. Carrington would refer to the treatise of Professor Love, mentioned in the introduction to the Paper, he would find a full explanation of his difficulties in regard to the relations between slopes and stresses, and torques and volumes.

The remarks communicated by Dr. Porter served to emphasize the fact that there was no connexion whatever between soap films and torsion, save through the medium of the ideal mathematical equations, and their application to practical problems must, of course, be made with as much reserve as was used, for instance, in applying the mathematical theory of beams and struts to the design of a bridge.



SOAP FILMS FOR TORSION PROBLEMS.